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Stability and stabilizability of mixed retarded-neutral type systems

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Abstract

We analyze the stability and stabilizability properties of mixed retarded-neutral type systems when the neutral term is allowed to be singular. Considering an operator model of the system in a Hilbert space we are interesting in the critical case when there exists a sequence of eigenvalues with real parts approaching to zero. In this case the exponential stability is not possible and we are studying the strong asymptotic stability property. The behavior of spectra of mixed retarded-neutral type systems does not allow to apply directly neither methods of retarded system nor the approach of neutral type systems for analysis of stability. In this paper two technics are combined to get the conditions of asymptotic non-exponential stability: the existence of a Riesz basis of invariant finite-dimensional subspaces and the boundedness of the resolvent in some subspaces of a special decomposition of the state space. For unstable systems the technics introduced allow to analyze the concept of regular strong stabilizability for mixed retarded-neutral type systems. The present paper extends the results on stability obtained in [R. Rabah, G.M. Sklyar, A. V. Rezounenko, Stability analysis of neutral type systems in Hilbert space. J. of Differential Equations, 214(2005), No. 2, 391–428] and the results on stabilizability from [R. Rabah, G.M. Sklyar, A. V. Rezounenko, On strong regular stabilizability for linear neutral type systems. J. of Differential Equations, 245(2008), No. 3, 569–593]. Comparing with the mentioned papers, here we avoid a restrictive assumption of non-singularity of the main neutral term.

Keywords. Retarded-neutral type systems, asymptotic non-exponential stability, stabilizability, infinite dimensional systems.

Mathematical subject classification. 93C23, 34K06, 34K20, 34K40, 49K25.

1 Introduction

The interest in considering delay differential equations and corresponding infinite-dimensional dynamical systems is caused by a huge amount of applied problem which can be described by these equations. The stability theory of such type of systems was studied intensively (see e.g. [2, 6, 11]). Number of results was obtained for retarded systems, however an analysis of neutral type systems is much more complicated and these systems

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are still studied not so deeply. We consider neutral type systems given by the following functional differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}[z(t) - Kz_t] = Lz_t + Bu(t), \qquad t \geq 0, \tag{1.1}$$

where $z_t : [-1, 0] \to \mathbb{C}^n$ is the history of z defined by $z_t(\theta) = z(t + \theta)$. We assume the delay operator $L : H^1([-1, 0], \mathbb{C}^n) \to \mathbb{C}^n$ to be linear and bounded, thus, it has the following form:

$$Lf = \int_{-1}^{0} A_2(\theta) f'(\theta)\, \mathrm{d}\theta + \int_{-1}^{0} A_3(\theta) f(\theta)\, \mathrm{d}\theta, \qquad f \in H^1([-1, 0], \mathbb{C}^n), \tag{1.2}$$

where A_2, A_3 are $n \times n$-matrices whose elements belong to $L_2([-1, 0], \mathbb{C})$. We take the difference operator K in the form

$$Kf = A_{-1} f(-1), \tag{1.3}$$

where A_{-1} is a constant $n \times n$-matrix. The form (1.3) may be considered as a particular case of the operator $K : C([-1, 0], \mathbb{C}^n) \to \mathbb{C}^n$ given by $Kf = \int_{-1}^{0} \mathrm{d}\mu(\theta) f(\theta)$, where $\mu(\cdot) : [-1, 0] \to \mathbb{C}^{n \times n}$ is of bounded variation and continuous at zero. However, the considered model (1.3) is sufficiently general. Its analysis is difficult enough and the results obtained are derived, in part, from the properties of the matrix A_{-1}.

The well-known approach, when studying systems of the form (1.1), is to consider a corresponding infinite-dimensional model $\dot{x} = \mathcal{A}x$, where $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup. For systems (1.1)–(1.3) the resolvent of the operator $\mathcal{A}$ allows an explicit representation (see [20, 21]). Such a representation is an effective tool for analyzing the exponential stability property since the last is equivalent to the uniform boundedness of the resolvent on the complex right half-plane. The resolvent boundedness approach is exhaustive when one considers stability of pure retarded type systems ($A_{-1} = 0$) since such systems may be exponentially stable or unstable only. This fact is due to that exponential growth of the semigroup $\{e^{t\mathcal{A}}\}_{t \geq 0}$ is determinated by spectrum's location and there are only a finite number of eigenvalues of $\mathcal{A}$ in any half-plane $\{\lambda : \operatorname{Re}\lambda \geq C\}$.

For neutral-type systems ($A_{-1} \neq 0$) in addition to the notion of exponential stability, which is characterized by the condition that the spectrum is bounded away from the imaginary axis (see [8, Theorem 6.1], [6]), one meets the notion of *strong* asymptotic non-exponential stability. This type of stability may happen in some critical case when the exponential stability is not possible (see e.g. [3]). Thus, strong stability cannot be described in terms of the resolvent boundedness. In [20, 21] for neutral type systems with a nonsingular neutral term ($\det A_{-1} \neq 0$) this type of stability was precisely investigated for systems of the form (1.1)–(1.3) and some necessary and sufficient conditions of strong stability and instability had been proved. The proofs are based on such a powerful tool as existence of a Riesz basis of $\mathcal{A}$-invariant finite-dimensional subspaces of the state space and on further application of the results on strong stability in Banach spaces that had been originated in [25] and later developed in [1, 13, 23, 24] and many others (see e.g. [27] for a review).

In the case of neutral type systems with a singular neutral term ($\det A_{-1} = 0$ and $A_{-1} \neq 0$), which we call *mixed retarded-neutral* type systems, the strong stability may also happen. However, the approach given in [20, 21] cannot be directly applied to such systems, since the existence of a Riesz basis of $\mathcal{A}$-invariant finite-dimensional subspaces of the whole state space cannot be guarantied. Moreover, mixed retarded-neutral type systems, in general, cannot be decomposed onto systems of pure neutral and pure retarded types. Therefore, the

analysis of strong stability for mixed retarded-neutral type systems poses a hard problem which requires bringing in essential ideas in addition.

The method presented in this paper is based on a decomposition of the initial infinite-dimensional model $\dot{x} = \mathcal{A}x$ onto two systems $\dot{x}_0 = \mathcal{A}_0 x_0$ and $\dot{x}_1 = \mathcal{A}_1 x_1$ in such a way that the spectra of $\mathcal{A}_0$, $\mathcal{A}_1$ satisfy: $\mathrm{Re}\,\sigma(\mathcal{A}_0) \leq -\varepsilon$ and $-\varepsilon < \mathrm{Re}\,\sigma(\mathcal{A}_1) < 0$ for some $\varepsilon > 0$. Generally speaking, the operators $\mathcal{A}_0$ and $\mathcal{A}_1$ are not the models of delay systems (retarded or neutral), what, in particular, implies that the relation between their exponential growth and spectrum's location is unknown a priori. We prove the exponential stability of the operator $\mathcal{A}_0$ analyzing the boundedness of its resolvent. This direct analysis requires subtle estimates and the proof is technically complicated. For the analysis of the subsystem $\dot{x}_1 = \mathcal{A}_1 x_1$ we apply methods of strong stability introduced in [20, 21]. Finally, the introduced approach allows us to prove for mixed retarded-neutral type systems the results on strong stability formulated in [21].

Besides, for control systems the proposed approach allows to analyze the notion of *regular* asymptotic stabilizability [22] which is closely related to the strong stability notion. The technic of the regular asymptotic stabilizability were introduced in [22] and the sufficient condition for the system (1.1)–(1.3) to be stabilizable had been proved in the case $\det A_{-1} \neq 0$. In the present paper, using the same framework as for stability, we show that these results hold for mixed retarded-neutral type systems also.

The general framework which we use is the theory of C_0-semigroups of linear bounded operators (see e.g. [27]). In order to precise the main contribution of our paper let us first give the operator model of the system (1.1)–(1.3). We use the model introduced by Burns et al. [4] in a Hilbert state space. The state operator is given by

$$\mathcal{A}x(t) = \mathcal{A}\left(\begin{array}{c} y(t) \\ z_t(\cdot) \end{array} \right) = \left(\begin{array}{c} \int_{-1}^0 A_2(\theta)\dot{z}_t(\theta)\ \mathrm{d}\theta + \int_{-1}^0 A_3(\theta)z_t(\theta)\ \mathrm{d}\theta \\ \mathrm{d}z_t(\theta)/\ \mathrm{d}\theta \end{array} \right), \qquad (1.4)$$

with the domain

$$\mathcal{D}(\mathcal{A}) = \{(y, z(\cdot))^T : \ z \in H^1(-1, 0; \mathbb{C}^n), y = z(0) - A_{-1}z(-1)\} \subset M_2, \qquad (1.5)$$

where $M_2 \overset{\mathrm{def}}{=} \mathbb{C}^n \times L_2(-1, 0; \mathbb{C}^n)$ is the state space. The operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup. Studying stability problem, we consider the model

$$\dot{x} = \mathcal{A}x, \qquad x(t) = \left(\begin{array}{c} y(t) \\ z_t(\cdot) \end{array} \right), \qquad (1.6)$$

corresponding to the equation (1.1)–(1.3) with the control $u \equiv 0$, i.e. to the equation

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + \int_{-1}^0 A_2(\theta)\dot{z}(t+\theta)\ \mathrm{d}\theta + \int_{-1}^0 A_3(\theta)z(t+\theta)\ \mathrm{d}\theta, \qquad t \geq 0. \qquad (1.7)$$

The solutions of (1.7) and (1.6) are related as $z_t(\theta) = z(t+\theta)$, $\theta \in [-1, 0]$.

For the control the equation (1.1)–(1.3) which we rewrite as

$$\dot{z}(t) = A_{-1}\dot{z}(t-1) + \int_{-1}^0 A_2(\theta)\dot{z}(t+\theta)\ \mathrm{d}\theta + \int_{-1}^0 A_3(\theta)z(t+\theta)\ \mathrm{d}\theta + Bu, \qquad (1.8)$$

we consider the model

$$\dot{x} = \mathcal{A}x + \mathcal{B}u, \qquad (1.9)$$

where the operator $\mathcal{B} : \mathbb{C}^p \to M_2$ is defined by $n \times p$-matrix B as follows: $\mathcal{B}u \overset{\mathrm{def}}{=} (Bu,\ 0)^T$.

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The operator $\mathcal{A}$ given by (1.4) possesses only discrete spectrum $\sigma(\mathcal{A})$, and, moreover, the growth of the semigroup $\{e^{t\mathcal{A}}\}_{t\geq 0}$ is determinated by spectrum's location. Namely, denoting by $\omega_s = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(\mathcal{A})\}$ and by $\omega_0 = \inf\{\omega : \|e^{\mathcal{A}t}x\| \leq Me^{\omega t}\|x\|\}$, we have the relation $\omega_0 = \omega_s$ (see e.g. [6]).

For stability problem, the last fact implies that the semigroup $\{e^{t\mathcal{A}}\}_{t\geq 0}$ is exponentially stable if and only if the spectrum of $\mathcal{A}$ satisfies $\omega_s < 0$. However, this type of stability is not the only possible one for systems of the form (1.7) (e.g. the same situation can also happen for some hyperbolic partial differential equation). Namely, if $\omega_s = 0$ (and $A_{-1} \neq 0$), then there exists a sequence of eigenvalues with real parts approaching to zero and imaginary part tending to infinity. In this critical case the exponential stability is not possible: $\|e^{t\mathcal{A}}\| \not\to 0$ when $t \to \infty$, but asymptotic non-exponential stability may occur: $\lim\limits_{t\to+\infty} e^{t\mathcal{A}}x = 0$ for all $x \in M_2$. For systems (1.6), satisfying the assumption $\det A_{-1} \neq 0$, the problem of strong stability was analyzed in [20, 21]. The main result on stability obtained there may be formulated as follows.

Theorem 1.1 ([21, R. Rabah, G.M. Sklyar, A.V. Rezounenko]). *Consider the system (1.6) such that $\det A_{-1} \neq 0$. Let us put $\sigma_1 = \sigma(A_{-1}) \cap \{\mu : |\mu| = 1\}$. Assume that $\sigma(\mathcal{A}) \subset \{\lambda : \operatorname{Re}\lambda < 0\}$ (necessary condition). The following three mutually exclusive possibilities hold true:*

(i) σ_1 consists of simple eigenvalues only, i.e. an one-dimensional eigenspace corresponds to each eigenvalue and there are no root vectors. Then system (1.6) is asymptotically stable.

(ii) The matrix A_{-1} has a Jordan block, corresponding to an eigenvalue $\mu \in \sigma_1$. Then (1.6) is unstable.

(ii) There are no Jordan blocks, corresponding to eigenvalues from σ_1, but there exists an eigenvalue $\mu \in \sigma_1$ whose eigenspace is at least two dimensional. In this case system (1.6) can be either stable or unstable. Moreover, there exist two systems with the same spectrum, such that one of them is stable while the other one is unstable.

Let us discuss the importance of the assumption $\det A_{-1} \neq 0$. The proof of Theorem 1.1 given in [21] is based on the following facts. Firstly, if $\det A_{-1} \neq 0$, then the spectrum of $\mathcal{A}$ is located in a vertical strip $d_1 \leq \operatorname{Re} \sigma(\mathcal{A}) \leq d_2$. Namely, in [20, 21] it was shown that $\sigma(\mathcal{A}) = \{\ln|\mu_m| + i(\arg\mu_m + 2\pi k) + \overline{o}(1/k) : \mu_m \in \sigma(A_{-1}), k \in \mathbb{Z}\}$. From the last it also follows the necessary condition for the system to be asymptotically stable: $\sigma(A_{-1}) \subset \{\mu : |\mu| \leq 1\}$.

Secondly, such location of the spectrum had allowed to prove the existence of a Riesz basis of generalized eigenvectors for the operator $\mathcal{A} = \widetilde{\mathcal{A}}$ corresponding to the case $A_2(\theta) \equiv A_3(\theta) \equiv 0$. For a general operator $\mathcal{A}$ the generalized eigenvectors may not constitute a basis of the state space (see an example in [20] and some general conditions in [28]). However, in [20, 21] it was proved the existence of a Riesz basis of $\mathcal{A}$-invariant finite-dimensional subspaces of the space M_2 (see also [29]). Such a basis is a powerful tool that had been applied for the analysis of strong stability.

If we allow the matrix A_{-1} to be singular, then the described above location of the spectrum does not hold anymore. Generally speaking, in this case for any $\alpha \in \mathbb{R}$ there exists an infinite number of eigenvalues which are situated on the left of the vertical line $\operatorname{Re}\lambda = \alpha$. Thus, the existence of a Riesz basis of $\mathcal{A}$-invariant finite-dimensional subspaces for the whole space M_2 cannot be guarantied. As a consequence, the proof of the item (i) given in [21], which is essentially based on the Riesz basis technic, is no longer satisfactory and one needs another way of the analysis of stability.

However, it can be asserted that nonzero $\mu_m \in \sigma(A_{-1})$ define the spectral set $\{\ln|\mu_m| + i(\arg\mu_m + 2\pi k) + \overline{o}(1/k) : \mu_m \in \sigma(A_{-1}), \mu_m \neq 0, k \in \mathbb{Z}\} \subset \sigma(\mathcal{A})$ which belongs to a

vertical strip. In particular, this can be asserted for $\mu_m \in \sigma_1$. The fact that Theorem 1.1 is formulated in terms of σ_1 and the last remark give us the idea to decompose the initial system (1.6) into two systems

$$\dot{x} = \mathcal{A}x \Leftrightarrow \begin{cases} \dot{x}_0 = \mathcal{A}_0 x_0 \\ \dot{x}_1 = \mathcal{A}_1 x_1 \end{cases} \tag{1.10}$$

in such way that $\sigma(\mathcal{A}_0) = \sigma(\mathcal{A}) \cap \{\lambda : -\infty < \mathrm{Re}\lambda \leq -\varepsilon\}$ and $\sigma(\mathcal{A}_1) = \sigma(\mathcal{A}) \cap \{\lambda : -\varepsilon < \mathrm{Re}\lambda \leq \omega_s = 0\}$, for some $\varepsilon > 0$.

To obtain the representation (1.10) we construct a special spectral decomposition of the state space: $M_2 = M_2^0 \oplus M_2^1$, where M_2^0, M_2^1 are $\mathcal{A}$-invariant subspaces. We define $\mathcal{A}_0 = \mathcal{A}|_{M_2^0}$ and $\mathcal{A}_1 = \mathcal{A}|_{M_2^1}$.

The spectrum of the system $\dot{x}_1 = \mathcal{A}_1 x_1$ is such that the corresponding eigenvectors form a Riesz basis of the subspace M_2^1. The strong stability of the semigroup $\{e^{t\mathcal{A}}|_{M_2^1}\}_{t \geq 0}$ is being proved using the methods of [21].

The semigroup $\{e^{t\mathcal{A}}|_{M_2^0}\}_{t \geq 0}$ is exponentially stable. We prove this fact using the equivalent condition consisting in the uniform boundedness of the resolvent $R(\lambda, \mathcal{A})|_{M_2^0}$ on the set $\{\lambda : \mathrm{Re}\lambda \geq 0\}$. Thus, we prove that the initial system $\dot{x} = \mathcal{A}x$ is asymptotically stable. The mentioned scheme requires complicated technics.

To complete the stability analysis we revisit the example showing the item (iii) with a simpler formulation than in [21], where it was given using the Riesz basis technic. The analysis of the spectrum being carried out in our example is essentially based on the deep results on transcendental equations obtained by L. Pontryagin [17]. We notice also that the proof of the item (ii) given in [21] does not involve the Riesz basis technic and, thus, it remains the same for the case $\det A_{-1} = 0$.

The technics of the direct spectral decompositions and the resolvent boundedness presented above allow us to extend the results on the stabilizability problem given in [22] for the case of singular matrix A_{-1}.

The general problem of stabilizability of control system is to find a feedback $u = \mathcal{F}x$ such that the closed-loop system

$$\dot{x} = (\mathcal{A} + \mathcal{B}\mathcal{F})x$$

is asymptotically stable in some sense. For the system (1.8) the result of exponential stabilizability may be derived from those obtained for some particular cases (see e.g. [7, 15, 16]). The needed feedback for our system is of the form

$$F(z(t + \cdot)) = F_{-1}\dot{z}(t - 1) + \int_{-1}^0 F_2(\theta)\dot{z}(t + \theta)\,\mathrm{d}\theta + \int_{-1}^0 F_3(\theta)z(t + \theta)\,\mathrm{d}\theta. \tag{1.11}$$

Our purpose is to obtain, as in [22], the condition of asymptotic non-exponential stabilizability of the system (1.8) with the regular feedback

$$F(z(t + \cdot)) = \int_{-1}^0 F_2(\theta)\dot{z}(t + \theta)\,\mathrm{d}\theta + \int_{-1}^0 F_3(\theta)z(t + \theta)\,\mathrm{d}\theta, \tag{1.12}$$

where $F_2(\cdot), F_3(\cdot) \in L_2(-1, 0; \mathbb{C}^{n \times p})$. The motivation is that this kind of feedback is relatively bounded with respect to the state operator $\mathcal{A}$ and does not change the domain of $\mathcal{A}$: $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A} + \mathcal{B}\mathcal{F})$. The natural necessary condition regular stabilizability is $\sigma(A_{-1}) \subset \{\mu : |\mu| \leq 1\}$ because A_{-1} is not modified by the feedback. Under the same restrictive condition $\det A_{-1} \neq 0$ in [22] was obtained the following result on stabilizability.

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Theorem 1.2 ([22, R. Rabah, G.M. Sklyar, A.V. Rezounenko]). *Let the system (1.8) verifies the following assumptions:*

    (1) *All the eigenvalues of the matrix A_{-1} satisfy $|\mu| \leq 1$.*

    (2) *All the eigenvalues $\mu \in \sigma_1$ are simple.*

Then the system is regularly asymptotically stabilizable if

    (3) $\mathrm{rank}(\triangle(\lambda),\ B) = n$ *for all* $\lambda : \mathrm{Re}\lambda \geq 0$.

    (4) $\mathrm{rank}(\mu I - A_{-1},\ B) = n$ *for all* $\mu \in \sigma_1$.

The proof of this theorem given in [22] uses the existence of the Riesz basis of the whole state space M_2 and, thus, it requires the assumption $\det A_{-1} \neq 0$. To avoid this assumption, we construct and prove another spectral decomposition which takes into account the unstable part of the system. By means of this decomposition we separate a subsystem which is generated by the part of the spectrum corresponding to the zero eigenvalues, i.e. the singularities of the matrix A_{-1}. Proving the resolvent boundedness, we show the exponential stability of this subsystem. The main "critical" part of the system is in $\mathcal{A}$-invariant subspaces, where we apply the same methods that were given in [22], namely, the theorem on infinite pole assignment, introduced there, and a classical pole assignment result in finite dimensional spaces.

The paper is organized as follows. In Sections 2 we recall the results on the spectrum, eigenvectors and the resolvent of the operator $\mathcal{A}$ obtained in [21, 22]. Besides we prove some properties of eigenvectors. In Section 3 we construct and prove two direct spectral decomposition of the state space. One of them is used to prove the main result on stability and another one for the proof the result on stabilizability. Section 4 is devoted to the proof of the uniform boundedness of the restriction of the resolvent on some invariant subspaces. Finally, in Section 5 and Section 6 we give the formulation and the proof of our main results on stability and stabilizability. Besides, in Section 5 we give an explicit example of two systems having the same spectrum in the open left half-plane but one of these systems is asymptotically stable while the other one is unstable.

2   Preliminaries

In this section we recall several results on the location of the spectrum of the operator $\mathcal{A}$, on the explicit form of its resolvent and on the form of eigenvectors of $\mathcal{A}$ and $\mathcal{A}^*$. We prove some properties of eigenvectors of $\mathcal{A}$ and $\mathcal{A}^*$.

2.1   The resolvent and the spectrum

The results given in this subsection have been presented and proved in [20, 21, 22]. Some formulations of the propositions are adapted for the case $\det A_{-1} = 0$.

Proposition 2.1 ([21, Proposition 1]). *The resolvent of the operator $\mathcal{A}$ has the following form:*

$$R(\lambda, \mathcal{A})\begin{pmatrix} z \\ \xi(\cdot) \end{pmatrix} \equiv \begin{pmatrix} \mathrm{e}^{-\lambda}A_{-1}\int_{-1}^{0}\mathrm{e}^{-\lambda s}\xi(s)\mathrm{d}s + (I - \mathrm{e}^{-\lambda}A_{-1})\triangle^{-1}(\lambda)D(z,\xi,\lambda) \\ \int_{0}^{\theta}\mathrm{e}^{\lambda(\theta-s)}\xi(s)\mathrm{d}s + \mathrm{e}^{\lambda\theta}\triangle^{-1}(\lambda)D(z,\xi,\lambda) \end{pmatrix}, \quad (2.13)$$

where $z \in \mathbb{C}^n, \xi(\cdot) \in L_2(-1,0;\mathbb{C}^n)$; $\triangle(\lambda)$ is the matrix function defined by

$$\triangle(\lambda) = \triangle_{\mathcal{A}}(\lambda) = -\lambda I + \lambda\mathrm{e}^{-\lambda}A_{-1} + \lambda\int_{-1}^{0}\mathrm{e}^{\lambda s}A_2(s)\mathrm{d}s + \int_{-1}^{0}\mathrm{e}^{\lambda s}A_3(s)\mathrm{d}s, \quad (2.14)$$

and $D(z, \xi, \lambda)$ is the following vector-function acting to $\mathbb{C}^n$:

$$D(z, \xi, \lambda) = z + \lambda e^{-\lambda} A_{-1} \int_{-1}^{0} e^{-\lambda\theta} \xi(\theta) \, d\theta - \int_{-1}^{0} A_2(\theta) \xi(\theta) \, d\theta$$

$$- \int_{-1}^{0} e^{\lambda\theta} [\lambda A_2(\theta) + A_3(\theta)] \left[\int_{0}^{\theta} e^{-\lambda s} \xi(s) \, ds \right] \, d\theta. \tag{2.15}$$

From (2.13) one may see that the resolvent does not exist in the points of singularity of the matrix $\triangle(\lambda)$, i.e. the equation $\det \triangle(\lambda) = 0$ defines the eigenvalues of the operator $\mathcal{A}$. Now let us characterize the spectrum of $\mathcal{A}$ more precisely.

We denote by $\mu_1, \ldots, \mu_\ell$ the set of distinct eigenvalues of the matrix A_{-1} and by $p_1, \ldots, p_\ell$ their multiplicities. We recall the notation $\sigma_1 = \sigma(A_{-1}) \cap \{\mu : |\mu| = 1\}$ and assume that $\sigma_1 = \{\mu_1, \ldots, \mu_{\ell_1}\}$, $\ell_1 \leq \ell$. We notice that one of the eigenvalues $\mu_{\ell_1+1}, \ldots, \mu_\ell$ may be zero.

Further, studying stability and stabilizability problems, we consider mainly the situations when the eigenvalues from σ_1 are simple. This gives us a motivation to assume below (if the opposite is not mentioned) that $p_1 = \ldots = p_{\ell_1} = 1$. Besides, without loss of generality, we assume that the matrix A_{-1} is in the following Jordan form:

$$A_{-1} = \begin{pmatrix} \mu_1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \mu_{\ell_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & J_{\ell_1+1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & J_\ell \end{pmatrix}, \tag{2.16}$$

where $J_{\ell_1+1}, \ldots, J_\ell$ are Jordan blocks corresponding to the eigenvalues $\mu_{\ell_1+1}, \ldots, \mu_\ell$.

Let us denote by $\widetilde{\mathcal{A}}$ the state operator in the case when $A_2(\theta) \equiv A_3(\theta) \equiv 0$. It is not difficult to see that the spectrum of $\widetilde{\mathcal{A}}$ has the following structure

$$\sigma(\widetilde{\mathcal{A}}) = \{\widetilde{\lambda}_m^k = \ln |\mu_m| + i(\arg \mu_m + 2\pi k) : \ m = 1, \ldots, \ell, \mu_m \neq 0, k \in \mathbb{Z}\} \cup \{0\}.$$

We denote by $L_m^k(r^{(k)})$ circles centered at $\widetilde{\lambda}_m^k$ with radii $r^{(k)}$.

Proposition 2.2. *Let $\sigma_1 = \{\mu_1, \ldots, \mu_{\ell_1}\}$ consists of simple eigenvalues only. There exists $N_1 \in \mathbb{N}$ such that the total multiplicity of the roots of the equation $\det \triangle(\lambda) = 0$, contained in the circles $L_m^k(r^{(k)})$, equals $p_m = 1$ for all $m = 1, \ldots, \ell_1$ and $k : |k| \geq N_1$, and the radii $r^{(k)}$ satisfy the relation $\sum_{k \in \mathbb{Z}} (r^{(k)})^2 < \infty$.*

This proposition is a particular case of [22, Theorem 4] which have been formulated and proved there under the assumption $\det A_{-1} \neq 0$. The proof for the case $\det A_{-1} = 0$ remains the same.

Notation 2.1. *We denote the eigenvalues of $\mathcal{A}$ mentioned in Proposition 2.2 by λ_m^k, $m = 1, \ldots, \ell_1$, $|k| \geq N_1$.*

Remark 2.3. *Proposition 2.2 is formulated for $m = 1, \ldots, \ell_1$, however, it also holds for all those indices $m = 1, \ldots, \ell$ which correspond to nonzero eigenvalues $\mu_m \in \sigma(A_{-1})$.*

Remark 2.4. *In the case $\det A_{-1} \neq 0$ the spectrum of $\mathcal{A}$ belongs to a vertical strip which is bounded from the left and from the right. However, in the case $\det A_{-1} = 0$, in addition to the eigenvalues mentioned in Proposition 2.2, the operator $\mathcal{A}$ may also possess an infinite sequence of eigenvalues with real parts tending to $-\infty$.*

Similar results hold for the operator $\mathcal{A}^*$. The spectra of $\mathcal{A}$ and $\mathcal{A}^*$ are related as $\sigma(\mathcal{A}^*) = \overline{\sigma(\mathcal{A})}$. Eigenvalues of $\mathcal{A}^*$ are the roots of the equation $\det \triangle^*(\lambda) = 0$, where

$$\triangle^*(\lambda) = \triangle_{\mathcal{A}^*}(\lambda) = -\lambda I + \lambda e^{-\lambda} A_{-1}^* + \lambda \int_{-1}^{0} e^{\lambda s} A_2^*(s) \, ds + \int_{-1}^{0} e^{\lambda s} A_3^*(s) \, ds, \tag{2.17}$$

and the relation $(\triangle(\lambda))^* = \triangle^*(\overline{\lambda})$ holds. The eigenvalues $\overline{\lambda_m^k}$, $m = 1, \ldots, \ell_1$, $|k| \geq N_1$ may be described as in Proposition 2.2.

2.2 Eigenvectors of $\mathcal{A}$ and $\mathcal{A}^*$

First we give the explicit form of eigenvectors which has been proved in [21, 22].

Proposition 2.5 ([21, Theorem 2], [22, Theorem 7]). *Eigenvectors φ: $(\mathcal{A} - \lambda I)\varphi = 0$ and ψ: $(\mathcal{A}^* - \overline{\lambda} I)\psi = 0$, $\lambda \in \sigma(\mathcal{A})$ are of the form:*

$$\varphi = \varphi(\lambda) = \begin{pmatrix} (I - e^{-\lambda} A_{-1}) x \\ e^{\lambda \theta} x \end{pmatrix}, \tag{2.18}$$

$$\psi = \psi(\overline{\lambda}) = \begin{pmatrix} y \\ \left[\overline{\lambda} e^{-\overline{\lambda}\theta} - A_2^*(\theta) + e^{-\overline{\lambda}\theta} \int_0^\theta e^{\overline{\lambda} s} (A_3^*(s) + \overline{\lambda} A_2^*(s)) ds \right] y \end{pmatrix}, \tag{2.19}$$

where $x = x(\lambda) \in \mathrm{Ker}\triangle(\lambda)$, $y = y(\overline{\lambda}) \in \mathrm{Ker}\triangle^*(\overline{\lambda})$.

Below we give several properties of the sets of eigenvectors and we begin with the calculation of the scalar product between eigenvectors of $\mathcal{A}$ and $\mathcal{A}^*$.

Lemma 2.6. *Let $\lambda_0, \lambda_1 \in \sigma(\mathcal{A})$ and $\varphi = \varphi(\lambda_0)$, $\psi = \psi(\overline{\lambda_1})$ are corresponding eigenvectors: $(\mathcal{A} - \lambda_0 I)\varphi = 0$, $(\mathcal{A}^* - \overline{\lambda_1} I)\psi = 0$. The scalar product $\langle \varphi, \psi \rangle_{M_2}$ equals to the following value:*

$$\langle \varphi, \psi \rangle_{M_2} = \begin{cases} 0, & \lambda_0 \neq \lambda_1 \\ -\langle \triangle'(\lambda_0) x, y \rangle_{\mathbb{C}^n}, & \lambda_0 = \lambda_1 \end{cases}, \tag{2.20}$$

where $\triangle'(\lambda) = \frac{d}{d\lambda} \triangle(\lambda)$ and $x = x(\lambda_0)$, $y = y(\overline{\lambda_1})$ are defined by (2.18) and (2.19).

Proof. First let $\lambda_0 \neq \lambda_1$ and we compute directly the scalar product $\langle \varphi, \psi \rangle_{M_2}$ using the representations (2.18) and (2.19):

$$\begin{aligned}
\langle \varphi, \psi \rangle_{M_2} &= \langle (I - e^{-\lambda_0} A_{-1}) x, y \rangle_{\mathbb{C}^n} + \int_{-1}^{0} \langle e^{\lambda_0 \theta} x, \overline{\lambda_1} e^{-\overline{\lambda_1}\theta} y \rangle_{\mathbb{C}^n} \, d\theta - \int_{-1}^{0} \langle e^{\lambda_0 \theta} x, A_2^*(\theta) y \rangle_{\mathbb{C}^n} \, d\theta \\
&\quad + \int_{-1}^{0} \left\langle e^{\lambda_0 \theta} x, e^{-\overline{\lambda_1}\theta} \int_0^\theta e^{\overline{\lambda_1} s} (A_3^*(s) + \overline{\lambda_1} A_2^*(s)) \, ds \cdot y \right\rangle_{\mathbb{C}^n} d\theta \\
&= \langle (I - e^{-\lambda_0} A_{-1}) x, y \rangle + \left\langle \int_{-1}^{0} \lambda_1 e^{(\lambda_0 - \lambda_1)\theta} \, d\theta \cdot x, y \right\rangle - \left\langle \int_{-1}^{0} e^{\lambda_0 \theta} A_2(\theta) \, d\theta \cdot x, y \right\rangle \\
&\quad + \left\langle \int_{-1}^{0} e^{(\lambda_0 - \lambda_1)\theta} \int_0^\theta e^{\lambda_1 s} (A_3(s) + \lambda_1 A_2(s)) \, ds \, d\theta \cdot x, \, y \right\rangle \\
&= \langle \Gamma(\lambda_0, \lambda_1) x, \, y \rangle,
\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}
\Gamma(\lambda_0, \lambda_1) &= I - e^{-\lambda_0} A_{-1} + \lambda_1 \int_{-1}^{0} e^{(\lambda_0 - \lambda_1)\theta} \, d\theta - \int_{-1}^{0} e^{\lambda_0 \theta} A_2(\theta) \, d\theta \\
&\quad + \int_{-1}^{0} e^{(\lambda_0 - \lambda_1)\theta} \int_0^\theta e^{\lambda_1 s} (A_3(s) + \lambda_1 A_2(s)) \, ds \, d\theta.
\end{aligned} \tag{2.22}$$

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The last term of $\Gamma(\lambda_0, \lambda_1)$, which is the integral over the domain $-1 \leq \theta \leq s \leq 0$, we rewrite using the identity $\int_{-1}^{0} \int_{0}^{\theta} G(s, \theta) \, ds \, d\theta = -\int_{-1}^{0} \int_{-1}^{s} G(s, \theta) \, d\theta \, ds$ which holds for any function $G(s, \theta)$. Taking into account the relation $\int_{-1}^{0} e^{(\lambda_0 - \lambda_1)\theta} \, d\theta = \frac{1}{\lambda_0 - \lambda_1}(1 - e^{\lambda_1 - \lambda_0})$, we obtain

$$\int_{-1}^{0} e^{(\lambda_0 - \lambda_1)\theta} \int_{0}^{\theta} e^{\lambda_1 s}(A_3(s) + \lambda_1 A_2(s)) \, ds \, d\theta = -\int_{-1}^{0} e^{\lambda_1 s}(A_3(s) + \lambda_1 A_2(s)) \int_{-1}^{s} e^{(\lambda_0 - \lambda_1)\theta} \, d\theta \, ds$$

$$= \tfrac{1}{\lambda_0 - \lambda_1}\left[e^{\lambda_1 - \lambda_0} \int_{-1}^{0} e^{\lambda_1 s}(A_3(s) + \lambda_1 A_2(s)) \, ds - \int_{-1}^{0} e^{\lambda_0 s}(A_3(s) + \lambda_1 A_2(s)) \, ds \right].$$

Finally, we have

$$\begin{aligned}
\Gamma(\lambda_0, \lambda_1) &= \tfrac{1}{\lambda_0 - \lambda_1}\Big[(\lambda_0 - \lambda_1)I - (\lambda_0 - \lambda_1)e^{-\lambda_0}A_{-1} + \lambda_1(1 - e^{\lambda_1 - \lambda_0})I \\
&\quad - (\lambda_0 - \lambda_1)\int_{-1}^{0} e^{\lambda_0 \theta}A_2(\theta) \, d\theta - \int_{-1}^{0} e^{\lambda_0 s}(A_3(s) + \lambda_1 A_2(s)) \, ds \\
&\quad + e^{\lambda_1 - \lambda_0}\int_{-1}^{0} e^{\lambda_1 s}(A_3(s) + \lambda_1 A_2(s)) \, ds \Big] \\
&= \tfrac{1}{\lambda_0 - \lambda_1}\Big[\lambda_0 I - \lambda_0 e^{-\lambda_0}A_{-1} - \int_{-1}^{0} e^{\lambda_0 s}(A_3(s) + \lambda_0 A_2(s)) \, ds \\
&\quad - \lambda_1 e^{\lambda_1 - \lambda_0}I + \lambda_1 e^{-\lambda_0}A_{-1} + e^{\lambda_1 - \lambda_0}\int_{-1}^{0} e^{\lambda_1 s}(A_3(s) + \lambda_1 A_2(s)) \, ds \Big] \\
&= \tfrac{1}{\lambda_0 - \lambda_1}\left[-\triangle(\lambda_0) + e^{\lambda_1 - \lambda_0}\triangle(\lambda_1) \right].
\end{aligned}$$

Taking into account that $x \in \mathrm{Ker}\triangle(\lambda_0)$, $y \in \mathrm{Ker}\triangle^*(\overline{\lambda_1})$ and $(\triangle(\lambda_1))^* = \triangle^*(\overline{\lambda_1})$, we conclude that

$$\langle \varphi, \psi \rangle_{M_2} = \left\langle \frac{1}{\lambda_0 - \lambda_1}\left[-\triangle(\lambda_0) + e^{\lambda_1 - \lambda_0}\triangle(\lambda_1) \right] x, y \right\rangle_{\mathbb{C}^n} = \frac{e^{\lambda_1 - \lambda_0}}{\lambda_0 - \lambda_1}\langle x, \triangle^*(\overline{\lambda_1})y \rangle_{\mathbb{C}^n} = 0.$$

$$(2.23)$$

Let us now consider the case $\lambda_0 = \lambda_1$. From (2.21), (2.22) we have:

$$\langle \varphi, \psi \rangle_{M_2} = \langle \Gamma(\lambda_0)x, y \rangle_{\mathbb{C}^n},$$

where

$$\Gamma(\lambda_0) = I - e^{-\lambda_0}A_{-1} + \lambda_0 I - \int_{-1}^{0} e^{\lambda_0 \theta}A_2(\theta) \, d\theta + \int_{-1}^{0} \int_{0}^{\theta} e^{\lambda_0 s}(A_3(s) + \lambda_0 A_2(s)) \, ds \, d\theta. \quad (2.24)$$

The last term of $\Gamma(\lambda_0)$, which is the integral over the domain $-1 \leq \theta \leq s \leq 0$, we rewrite using the identity $\int_{-1}^{0} \int_{0}^{\theta} G(s, \theta) \, ds \, d\theta = -\int_{-1}^{0} \int_{-1}^{s} G(s, \theta) \, d\theta \, ds$. Thus, we obtain:

$$\begin{aligned}
\Gamma(\lambda_0) &= I - e^{-\lambda_0}A_{-1} + \lambda_0 I - \int_{-1}^{0} e^{\lambda_0 \theta}A_2(\theta) \, d\theta - \int_{-1}^{0} e^{\lambda_0 s}(A_3(s) + \lambda_0 A_2(s)) \int_{-1}^{s} d\theta \, ds \\
&= \left(I - e^{-\lambda_0}A_{-1} - \int_{-1}^{0} e^{\lambda_0 s}(sA_3(s) + s\lambda_0 A_2(s) + A_2(s)) \, ds \right) \\
&\quad + \left(\lambda_0 I - \int_{-1}^{0} e^{\lambda_0 s}(A_3(s) + \lambda_0 A_2(s)) \, ds \right) \\
&= -\triangle'(\lambda_0) - \triangle(\lambda_0).
\end{aligned}$$

Taking into account the relation $x \in \mathrm{Ker}\triangle(\lambda_0)$, we conclude that

$$\langle \varphi, \psi \rangle_{M_2} = -\langle \triangle'(\lambda_0)x, y \rangle_{\mathbb{C}^n}. \quad (2.25)$$

The last completes the proof of the lemma. □

For $\varphi(\lambda_m^k)$ and $\psi(\overline{\lambda_m^k})$ we will use the notation φ_m^k and ψ_m^k respectively. Besides, we use x_m^k and y_m^k instead of $x(\lambda_m^k)$ and $y(\overline{\lambda_m^k})$.

Lemma 2.7. *Let $\sigma_1 = \{\mu_1, \ldots, \mu_{\ell_1}\}$ consists of simple eigenvalues only. The eigenvectors φ_m^k, $m = 1, \ldots, \ell_1$, $k : |k| \geq N_1$ constitute a Riesz basis of the closure of their linear span. The same holds for eigenvectors ψ_m^k, $m = 1, \ldots, \ell_1$, $k : |k| \geq N_1$.*

A more general formulation of this proposition have been given in [20, Theorem 7, Theorem 15] under the assumption $\det A_{-1} \neq 0$. We give a sketch of the proof in our case.

The families of functions $\{e^{\widetilde{\lambda}_m^k \theta}\}_{k \in \mathbb{Z}}$ form an orthogonal basis of the space $L_2([-1,0], \mathbb{C})$ for each $m = 1, \ldots, \ell_1$, where $\widetilde{\lambda}_m^k = i(\arg\mu_m + 2\pi k)$ are eigenvalues of the operator $\widetilde{\mathcal{A}}$. Thus, the functions $\{e^{\widetilde{\lambda}_m^k \theta}\}_{|k| \geq N}$, $N \in \mathbb{N}$ form a basis of the closure of their linear span.

Since we have chosen the matrix A_{-1} in the form (2.16) and due to (2.18), the eigenvectors $\widetilde{\varphi}_m^k$ of $\widetilde{\mathcal{A}}$ are of the form $\widetilde{\varphi}_m^k = \begin{pmatrix} 0 \\ e^{\widetilde{\lambda}_m^k \theta} e_m \end{pmatrix}$, $e_m = (0, \ldots, 1, \ldots, 0)^T$. Therefore, the family $\{\widetilde{\varphi}_m^k : m = 1, \ldots, \ell_1 : |k| \geq N_1\}$ is a basis of the closure of its linear span.

The eigenvectors $\varphi_m^k = \begin{pmatrix} (I - e^{-\lambda_m^k} A_{-1}) x_m^k \\ e^{\lambda_m^k \theta} x_m^k \end{pmatrix}$ of $\mathcal{A}$ are quadratically close to $\widetilde{\varphi}_m^k$. To prove this fact we should argue similar to Theorem 15 given in [20] (see also [10]). Thus, eigenvectors φ_m^k, $m = 1, \ldots, \ell_1$, $k : |k| \geq N_1$ constitute a Riesz basis of the closure of their linear span.

From Lemma 2.6 and Lemma 2.7 we conclude the following.

Corollary 2.8. *The sequences φ_m^k and ψ_m^k, $m = 1, \ldots, \ell_1$, $k : |k| \geq N_1$ are biorthogonal after normalization and $\langle \varphi_m^k, \psi_m^k \rangle_{M_2} = -\langle \triangle'(\lambda_m^k) x_m^k, y_m^k \rangle_{\mathbb{C}^n}$.*

The following relation will be essentially used in the analysis of the boundedness of the resolvent in Section 4.

Lemma 2.9. *Let $\psi = \psi(\overline{\lambda_0})$, $\lambda_0 \in \sigma(\mathcal{A})$ be an eigenvector of the operator $\mathcal{A}^*$ and let $g = (z, \xi(\cdot)) \in M_2$ be orthogonal to ψ: $g \perp \psi$. Then the following relation holds:*

$$D(z, \xi, \lambda_0) \in \mathrm{Im}\triangle(\lambda_0), \tag{2.26}$$

where $D(z, \xi, \lambda)$ is defined by (2.1).

Proof. We show the relation $D(z, \xi, \lambda_0) \perp \mathrm{Ker}\triangle^*(\overline{\lambda_0})$ which is equivalent to (2.26). The eigenvector ψ is of the form (2.19):

$$\psi = \begin{pmatrix} y \\ \left[\overline{\lambda_0} e^{-\overline{\lambda_0}\theta} - A_2^*(\theta) + e^{-\overline{\lambda_0}\theta} \int_0^\theta e^{\overline{\lambda_0}s} A_3^*(s)\, ds + \overline{\lambda_0} e^{-\overline{\lambda_0}\theta} \int_0^\theta e^{\overline{\lambda_0}s} A_2^*(s)\, ds \right] y \end{pmatrix},$$

where $y = y(\overline{\lambda_0}) \in \mathrm{Ker}\triangle^*(\overline{\lambda_0})$. For any $g = (z, \xi(\cdot))$, which is orthogonal to ψ, we obtain:

$$\begin{aligned}
0 = \langle g, \psi \rangle_{M_2} &= \langle z, y \rangle_{\mathbb{C}^n} + \int_{-1}^0 \left\langle \xi(\theta), \overline{\lambda_0} e^{-\overline{\lambda_0}\theta} y \right\rangle_{\mathbb{C}^n} d\theta - \int_{-1}^0 \langle \xi(\theta), A_2^*(\theta) y \rangle_{\mathbb{C}^n}\, d\theta \\
&\quad + \int_{-1}^0 \left\langle \xi(\theta),\ e^{-\overline{\lambda_0}\theta} \int_0^\theta e^{\overline{\lambda_0}s} (A_3^*(s) + \overline{\lambda_0} A_2^*(s))\, ds \cdot y \right\rangle_{\mathbb{C}^n} d\theta \\
&= \langle z, y \rangle_{\mathbb{C}^n} + \left\langle \int_{-1}^0 \lambda_0 e^{-\lambda_0\theta} \xi(\theta)\, d\theta, y \right\rangle_{\mathbb{C}^n} - \left\langle \int_{-1}^0 A_2(\theta)\xi(\theta)\, d\theta, y \right\rangle_{\mathbb{C}^n} \\
&\quad + \left\langle \int_{-1}^0 \left[e^{-\lambda_0\theta} \int_0^\theta e^{\lambda_0 s} (A_3(s) + \lambda_0 A_2(s))\, ds \right] \xi(\theta)\, d\theta, y \right\rangle_{\mathbb{C}^n}.
\end{aligned} \tag{2.27}$$

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Using the identity $\int_{-1}^{0} \int_{0}^{\theta} G(s,\theta)\,\mathrm{d}s\,\mathrm{d}\theta = -\int_{-1}^{0} \int_{-1}^{s} G(s,\theta)\,\mathrm{d}\theta\,\mathrm{d}s$ which holds for any function $G(s,\theta)$, we rewrite the the last term of (2.27), and, finally, we obtain the relation:

$$
\begin{aligned}
0 = \langle g,\, \psi \rangle_{M_2} = \Big\langle & z + \int_{-1}^{0} \lambda_0 e^{-\lambda_0 \theta} \xi(\theta)\,\mathrm{d}\theta + \int_{-1}^{0} A_2(\theta)\xi(\theta)\,\mathrm{d}\theta \\
& - \int_{-1}^{0} e^{\lambda_0 s} \left[A_3(s) + \lambda_0 A_2(s)\right] \int_{-1}^{s} e^{-\lambda_0 \theta} \xi(\theta)\,\mathrm{d}\theta\,\mathrm{d}s,\, y \Big\rangle_{\mathbb{C}^n}.
\end{aligned}
\tag{2.28}
$$

Since $y \in \mathrm{Ker}\triangle^*(\overline{\lambda_0})$, then for any $x \in \mathbb{C}^n$ we have:

$$
0 = \langle x, \triangle^*(\overline{\lambda_0})y \rangle_{\mathbb{C}^n} = \langle \triangle(\lambda_0)x,\, y \rangle_{\mathbb{C}^n}.
$$

Therefore, for any θ the relation $\langle e^{-\lambda_0 \theta} \triangle(\lambda_0)\xi(\theta), y \rangle_{\mathbb{C}^n} = 0$ holds, and, integrating it by θ from -1 to 0, we obtain:

$$
\begin{aligned}
0 = \Big\langle \int_{-1}^{0} e^{-\lambda_0 \theta} \triangle(\lambda_0)\xi(\theta)\,\mathrm{d}\theta,\, y \Big\rangle = \Big\langle & -\int_{-1}^{0} \lambda_0 e^{-\lambda_0 \theta}\xi(\theta)\,\mathrm{d}\theta + \int_{-1}^{0} \lambda_0 e^{-\lambda_0 \theta} A_{-1}\xi(\theta)\,\mathrm{d}\theta \\
& + \int_{-1}^{0} e^{\lambda_0 s}\left[A_3(s) + \lambda_0 A_2(s)\right]\,\mathrm{d}s \int_{-1}^{0} e^{-\lambda_0 \theta}\xi(\theta)\,\mathrm{d}\theta,\, y \Big\rangle_{\mathbb{C}^n}.
\end{aligned}
\tag{2.29}
$$

Let us sum up the left-hand sides and the right-hand sides of the relations (2.29) and (2.28). In the obtained relation the term $\int_{-1}^{0} \lambda_0 e^{-\lambda_0 \theta}\xi(\theta)\,\mathrm{d}\theta$ is cancelled. The last terms of (2.29) and (2.28) we sum up according to the identity $-\int_{-1}^{0} \int_{-1}^{s} G(s,\theta)\,\mathrm{d}\theta\,\mathrm{d}s + \int_{-1}^{0} \int_{-1}^{0} G(s,\theta)\,\mathrm{d}\theta\,\mathrm{d}s = -\int_{-1}^{0} \int_{0}^{s} G(s,\theta)\,\mathrm{d}\theta\,\mathrm{d}s = -\int_{-1}^{0} \int_{0}^{\theta} G(\theta,s)\,\mathrm{d}s\,\mathrm{d}\theta$ which holds true for any function $G(s,\theta)$. Finally, we obtain:

$$
\begin{aligned}
0 &= \Big\langle z + \lambda_0 e^{-\lambda_0} A_{-1} \int_{-1}^{0} e^{-\lambda_0 \theta}\xi(\theta)\,\mathrm{d}\theta - \int_{-1}^{0} A_2(\theta)\xi(\theta)\,\mathrm{d}\theta \\
&\qquad - \int_{-1}^{0} e^{\lambda_0 \theta}\left[A_3(\theta) + \lambda_0 A_2(\theta)\right]\left[\int_{0}^{\theta} e^{-\lambda_0 s}\xi(s)\,\mathrm{d}s\right]\,\mathrm{d}\theta,\, y \Big\rangle_{\mathbb{C}^n} \\
&\equiv \langle D(z,\xi,\lambda_0),\, y \rangle_{\mathbb{C}^n}.
\end{aligned}
$$

Since $y \in \mathrm{Ker}\triangle^*(\overline{\lambda_0})$, we conclude that $D(z,\xi,\lambda_0) \perp \mathrm{Ker}\triangle^*(\overline{\lambda_0})$, what completes the proof of the lemma. $\qquad\square$

Remark 2.10. *We emphasize the fact that* $\det \triangle(\lambda_0) = 0$ *and, therefore, the matrix* $\triangle^{-1}(\lambda_0)$ *does not exist. However, the proved relation* $D(z,\xi,\lambda_0) \in \mathrm{Im}\triangle(\lambda_0)$ *means that there exists the inverse image of the vector* $D(z,\xi,\lambda_0)$ *with respect to the matrix* $\triangle(\lambda_0)$.

3 Spectral decompositions of the state space

We recall that we consider the operator $\mathcal{A}$ in the case when all eigenvalues from $\sigma_1 \subset \sigma(A_{-1})$ are simple. In this section we construct construct two spectral decompositions of the state space M_2. Assuming that $\sigma(\mathcal{A}) \subset \{\lambda : \mathrm{Re}\lambda < 0\}$, in the first subsection we construct a decomposition which we further us in Section 5 for the stability analysis. In the second subsection we assume only $|\mu| \leq 1$ for all $\mu \in \sigma(A_{-1})$ (i.e. a part of the spectrum of $\mathcal{A}$ may belongs to the closed right half-plane) and construct a decomposition needed in Section 6 for the stabilizability analysis. The structures of these decomposition are very similar. In the third subsection we prove some technical results used in the proofs of validity of the decompositions.

3.1 Spectral decomposition for the stability problem

For the stability analysis our aim is to divide the system onto exponentially stable part and strongly asymptotically stable part. To do this we construct a decomposition of the state space M_2 onto the direct sum of two $\mathcal{A}$-invariant subspaces and prove its validity.

We divide the spectrum of $\mathcal{A}$ onto two parts. For some $N \geq N_1$ we define

$$\Lambda_1 = \Lambda_1(N) = \{\lambda_m^k \in \sigma(\mathcal{A}), \ m = 1, \ldots, \ell_1, |k| \geq N\}, \tag{3.30}$$

and represent the spectrum as follows:

$$\sigma(\mathcal{A}) = \Lambda_0 \cup \Lambda_1.$$

Remark 3.1. *The set Λ_1 is determined by $N \in \mathbb{N}$. For any small $\varepsilon > 0$ there exists big enough N such that Λ_1 belongs to the vertical strip $\{\lambda : -\varepsilon < \mathrm{Re}\lambda < 0\}$.*

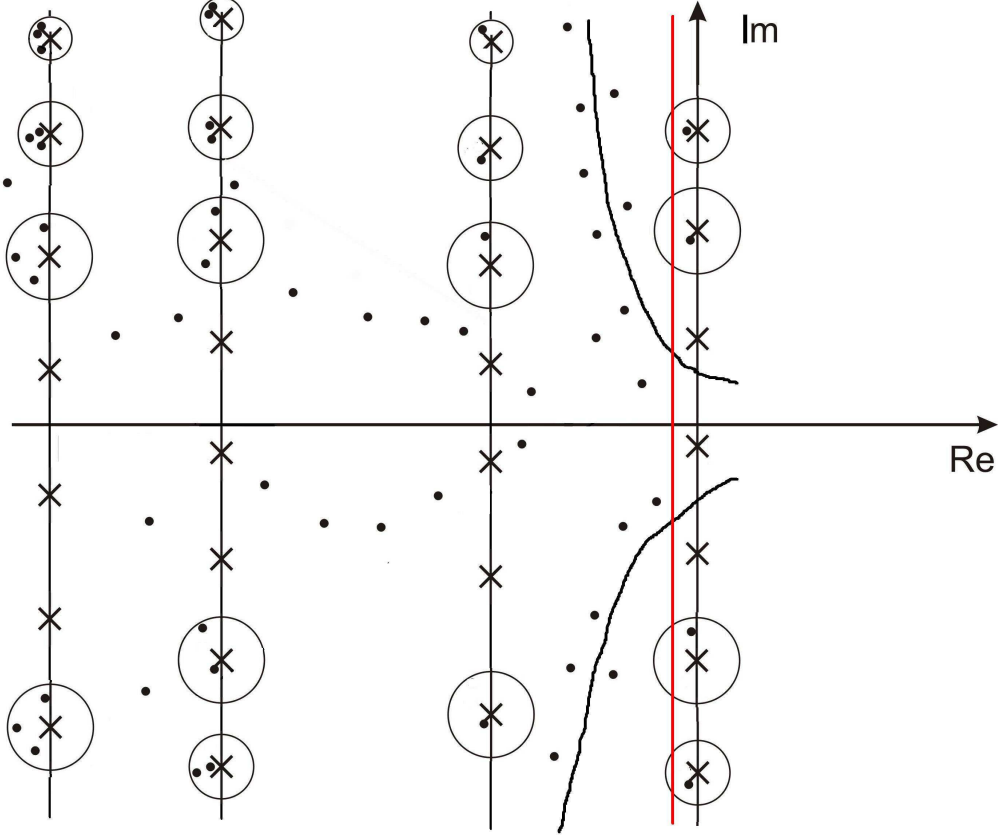The following figure illustrates our idea:

Figure 1.

By crosses we denote $\widetilde{\lambda}_m^k$, eigenvalues of $\widetilde{\mathcal{A}}$ and by points we denote eigenvalues of the operator $\mathcal{A}$.

We introduce two subspaces of M_2:

$$M_2^1 = M_2^1(N) = \mathrm{Cl}\,\mathrm{Lin}\{\varphi : \ (\mathcal{A} - \lambda I)\varphi = 0, \ \lambda \in \Lambda_1\}, \tag{3.31}$$

$$\widehat{M_2^1} = \widehat{M_2^1}(N) = \mathrm{Cl}\,\mathrm{Lin}\{\psi : \ (\mathcal{A}^* - \overline{\lambda}I)\psi = 0, \ \lambda \in \Lambda_1\}. \tag{3.32}$$

Obviously, M_2^1 is $\mathcal{A}$-invariant and $\widehat{M_2^1}$ is $\mathcal{A}^*$-invariant. We introduce $M_2^0 = M_2^0(N)$ which satisfies

$$M_2 = \widehat{M_2^1} \overset{\perp}{\oplus} M_2^0. \tag{3.33}$$

Due to the construction, M_2^0 is an $\mathcal{A}$-invariant subspace.

Remark 3.2. *We recall that due to Lemma 2.7 eigenvectors $\{\varphi_m^k\}$ of $\mathcal{A}$, corresponding to $\lambda_m^k \in \Lambda_1$, form a Riesz basis of the closure of their linear span. The same holds for eigenvectors $\{\psi_m^k\}$ of $\mathcal{A}^*$, corresponding to $\overline{\lambda_m^k}$, $\lambda_m^k \in \Lambda_1$.*

The main result of this subsection is the following theorem.

Theorem 3.3 (on direct decomposition). *Let $\sigma_1 = \{\mu_1, \ldots, \mu_{\ell_1}\}$ consists of simple eigenvalues only. For any $N \geq N_1$ the subset $\Lambda_1 = \Lambda_1(N) \subset \sigma(\mathcal{A})$ given by (3.30) and the subspaces M_2^0, M_2^1, $\widehat{M_2^1}$, given by (3.31), (3.32) and (3.33) define the direct decomposition of the space:*

$$M_2 = M_2^1 \oplus M_2^0 \tag{3.34}$$

where the subspaces M_2^1, M_2^0 are $\mathcal{A}$-invariant.

Proof. To prove (3.34) we show that any element $\xi \in M_2$ allows the following representation:

$$\xi = \xi_0 + \sum_{m=1}^{\ell_1} \sum_{|k|\geq N} c_m^k \varphi_m^k, \quad \xi_0 \in M_2^0, \ \varphi_m^k \in M_2^1, \ \sum_{m=1}^{\ell_1} \sum_{|k|\geq N} |c_m^k|^2 < \infty. \tag{3.35}$$

As we have noticed above, the eigenvectors $\{\varphi_m^k : (\mathcal{A} - \lambda_m^k I)\varphi_m^k = 0, \ \lambda_m^k \in \Lambda_1\}$ given by (2.18) form a Riesz basis of the closure of their linear span. Besides the eigenvectors $\{\frac{1}{\overline{\lambda_m^k}}\psi_m^k : (\mathcal{A}^* - \overline{\lambda_m^k}I)\psi_m^k = 0, \ \lambda_m^k \in \Lambda_1\}$ form a Riesz basis of the closure of their linear span, where eigenvectors ψ_m^k are given by (2.19). We use the notation $\widehat{\psi}_m^k = \frac{1}{\overline{\lambda_m^k}}\psi_m^k$.

Using the representations (2.18) and (2.19), we choose eigenvectors with $\|x_m^k\|_{\mathbb{C}^n} = 1$ and $\|y_m^k\|_{\mathbb{C}^n} = 1$. Due to Lemma 3.5 given in Subsection 3.3 there exists $C > 0$ such that $\|\varphi_m^k\|_{M^2} \leq C$ and $\|\widehat{\psi}_m^k\|_{M^2} \leq C$ for all $m = 1, \ldots, \ell_1$, $|k| \geq N$.

Applying the decomposition (3.33) to vectors φ_m^k, we obtain

$$\varphi_m^k = \gamma_m^k + \sum_{i=1}^{\ell_1} \sum_{|j|\geq N} a_i^j \widehat{\psi}_i^j, \quad \gamma_m^k \in M_2^0.$$

Since $\langle \varphi_m^k, \widehat{\psi}_i^j \rangle = 0$ for $(m, k) \neq (i, j)$ (Corollary 2.8), the last representation may be rewritten as follows:

$$\varphi_m^k = \gamma_m^k + a_m^k \widehat{\psi}_m^k, \quad \gamma_m^k \in M_2^0, \tag{3.36}$$

moreover, due to (2.20) we have the relation

$$a_m^k = \frac{\langle \varphi_m^k, \widehat{\psi}_m^k \rangle_{M_2}}{\|\widehat{\psi}_m^k\|_{M_2}^2} = \frac{\frac{1}{\lambda_m^k}\langle \varphi_m^k, \psi_m^k \rangle_{M_2}}{\|\widehat{\psi}_m^k\|_{M_2}^2} = \frac{-\frac{1}{\lambda_m^k}\langle \Delta'(\lambda_m^k)x_m^k, y_m^k \rangle_{\mathbb{C}^n}}{\|\widehat{\psi}_m^k\|_{M_2}^2}. \tag{3.37}$$

From (3.36) and (3.37) it also follows that

$$\|\gamma_m^k\| \leq \|\varphi_m^k\| + |a_m^k| \|\widehat{\psi}_m^k\| \leq C + \sqrt{|\frac{1}{\lambda_m^k}\langle \Delta'(\lambda_m^k)x_m^k, y_m^k \rangle|}. \tag{3.38}$$

Using the decomposition (3.33) and the relation (3.36), we represent each vector $\xi \in M_2$ as follows:

$$
\begin{aligned}
\xi &= \widehat{\xi}_0 + \sum_{m=1}^{\ell_1} \sum_{|k|\geq N} b_m^k \widehat{\psi}_m^k = \widehat{\xi}_0 - \sum_{m=1}^{\ell_1} \sum_{|k|\geq N} \frac{b_m^k}{a_m^k}\gamma_m^k + \sum_{m=1}^{\ell_1} \sum_{|k|\geq N} \frac{b_m^k}{a_m^k}\varphi_m^k \\
&= \xi_0 + \sum_{m=1}^{\ell_1} \sum_{|k|\geq N} c_m^k \varphi_m^k,
\end{aligned}
\tag{3.39}
$$

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where $\widehat{\xi}_0 \in M_2^0$, $\sum\limits_{m=1}^{\ell_1} \sum\limits_{|k|\geq N} |b_m^k|^2 < \infty$, $\xi_0 = \widehat{\xi}_0 - \sum\limits_{m=1}^{\ell_1} \sum\limits_{|k|\geq N} \frac{b_m^k}{a_m^k}\gamma_m^k \in M_2^0$, $c_m^k = \frac{b_m^k}{a_m^k}$.

To prove the validity of the decomposition (3.39) it is enough to show that $\left|\frac{1}{a_m^k}\right| \leq C_1$ and $\|\gamma_m^k\| \leq C_2$ for some $0 < C_1, C_2 < +\infty$. Taking into account (3.37) and (3.38), the last means to give the estimate

$$0 < C_1 \leq \left|\frac{1}{\lambda_m^k}\langle \Delta'(\lambda_m^k)x_m^k, y_m^k\rangle\right| \leq C_2, \qquad \lambda_m^k \in \Lambda_1. \tag{3.40}$$

This estimate is proved by Lemma 3.11 given in Subsection 3.3.

Thus, the representation (3.39) holds for any $\xi \in M_2$, what completes the proof of the theorem. $\qquad\square$

3.2 Spectral decomposition for the stabilizability problem

We recall that for the stabilizability problem we assume only that $|\mu| \leq 1$ for all $\mu \in \sigma(A_{-1})$. In this case an infinite number of eigenvalues may belong to the right-half plane. On the other hand, only a finite number of eigenvalues may be located on the right of a vertical line $\mathrm{Re}\lambda = \varepsilon$ for any $\varepsilon > 0$. For the analysis of stabilizability it is convenient to construct a decomposition of the state space onto three $\mathcal{A}$-invariant subspaces.

We divide the spectrum of $\mathcal{A}$ onto three parts:

$$\sigma(\mathcal{A}) = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2, \tag{3.41}$$

where the subsets Λ_0, Λ_1, Λ_2 are constructed by the following procedure.

Let N_0 be such that $\lambda_m^k \in L_m^k(r)$, $r \leq \frac{1}{3}|\widetilde{\lambda}_m^k - \widetilde{\lambda}_i^j|$, $(m,k) \neq (i,j)$ for all $k \geq N_0$ and for all m such that $\mu_m \neq 0$. First, we construct an auxiliary division

$$\sigma(\mathcal{A}) = \chi_1 \cup \chi_0, \qquad \chi_1 = \{\lambda_m^k \in \sigma(\mathcal{A}) : \ |k| \geq N_0, m = 1, \ldots, \ell, \mu_m \neq 0\}.$$

Due to the construction, any vertical strip $\mathrm{St}(\delta_1, \delta_2) = \{\lambda : \ \delta_1 < \mathrm{Re}\lambda < \delta_2\}$ contains only a finite number of eigenvalues from χ_0. We also recall that $\omega_s = \sup\{\mathrm{Re}\lambda : \ \lambda \in \sigma(\mathcal{A})\} < +\infty$.

If $\sigma_1 \neq \emptyset$ then for any $r > 0$ the strip $\mathrm{St}(-r, r)$ contains an infinite number of eigenvalues from χ_1 and, as we have noticed above, only a finite number of eigenvalues from χ_0. Let us fix some $r > 0$ and consider the value

$$\varepsilon = \min_{\lambda \in \mathrm{St}(-r,r)\cap\chi_0} |\mathrm{Re}\lambda|.$$

If $\varepsilon > 0$, then the vertical strip $\mathrm{St}(-\varepsilon, \varepsilon)$ does not contain eigenvalues from χ_0 and contains an infinite number of eigenvalues from χ_1. Moreover, the strip $\mathrm{St}(\varepsilon, r)$ contains only a finite number of eigenvalues from χ_1. Thus, the strip $\mathrm{St}(\varepsilon, \omega_s)$ contains a finite number of eigenvalues of the operator $\mathcal{A}$ and, therefore, we conclude that these eigenvalues are located in a rectangle $\{\lambda : \ \varepsilon \leq \mathrm{Re}\lambda \leq \omega_0, \ |\mathrm{Im}\lambda| < M\}$ for some $M > 0$. Finally, we put

$$\begin{aligned} \Lambda_0 &= \sigma(\mathcal{A}) \cap \{\lambda : \ \mathrm{Re}\lambda \leq -\varepsilon\}, \\ \Lambda_1 &= \sigma(\mathcal{A}) \cap \mathrm{St}(-\varepsilon, \varepsilon), \\ \Lambda_2 &= \sigma(\mathcal{A}) \cap \mathrm{St}(\varepsilon, \omega_s). \end{aligned} \tag{3.42}$$

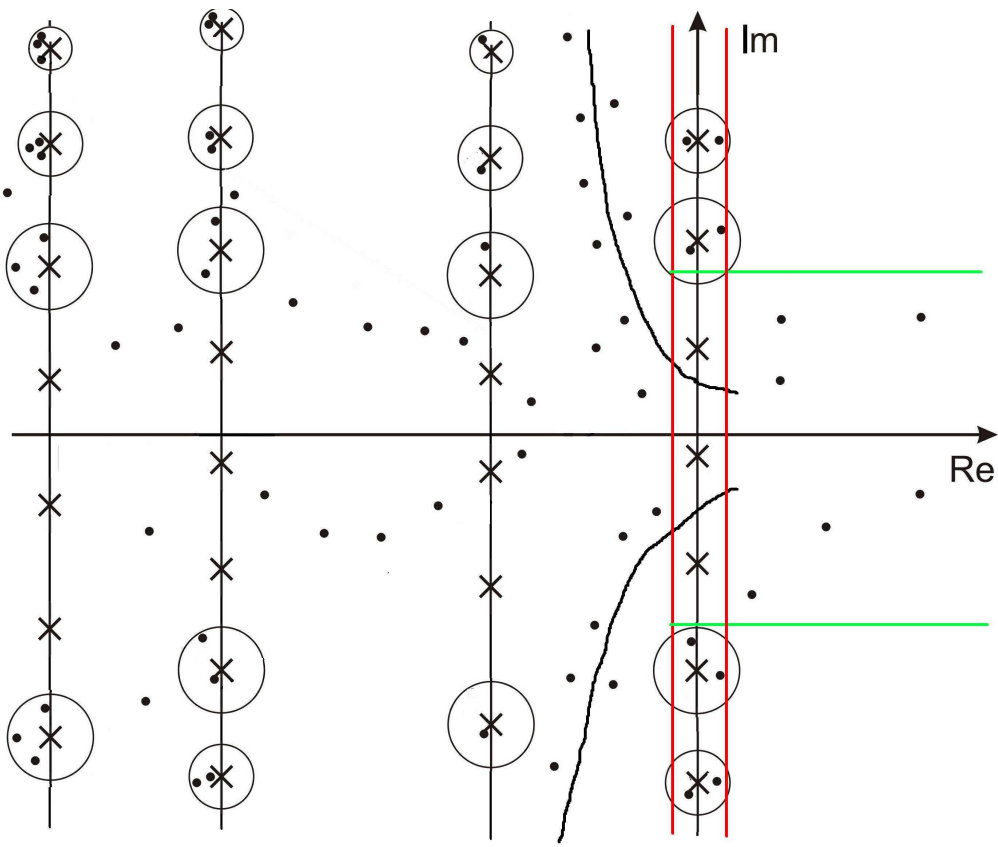We illustrate the mentioned construction by the following figure:

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Figure 2.

Again by crosses we denote $\widetilde{\lambda}_m^k$, eigenvalues of $\widetilde{\mathcal{A}}$ and by points we denote eigenvalues of the operator $\mathcal{A}$.

We notice that the relation $\varepsilon = 0$ means that there exists eigenvalues with zero real part. In these case we calculate $\min\limits_{\lambda \in \mathrm{St}(-r,r) \cap \chi_0} |\mathrm{Re}\lambda|$ without taking these eigenvalues into consideration and after constructing (3.42) we add these eigenvalues to Λ_2.

The obtained sets of eigenvalues may be described as follows: Λ_0 belongs to the left half-plane and is separated from the imaginary axis; Λ_1 consists of infinite number of simple eigenvalues which may be as stable as unstable, the corresponding eigenvectors form a Riesz basis of the closure of their linear span; Λ_2 consists of finite number of unstable eigenvalues.

Passing over to the construction of invariant subspaces, let us denote the elements of the finite set Λ_2 as λ_i, $i = 1, \ldots, r$. We denote the corresponding generalized eigenvectors by $\varphi_{i,j}$: $(\mathcal{A} - \lambda_i I)^j \varphi_{i,j} = 0$, $j = 0, \ldots, s_i - 1$. As before, the eigenvalues from Λ_1 we denote as λ_m^k and the corresponding eigenvectors we denote as φ_m^k, $m = 1, \ldots, \ell_1$, $|k| \geq N$.

We introduce the following two infinite-dimensional subspaces of eigenvectors:

$$M_2^1 = \mathrm{Cl}\,\mathrm{Lin}\{\varphi_m^k : \ (\mathcal{A} - \lambda_m^k I)\varphi_m^k = 0, \ \lambda_m^k \in \Lambda_1\},$$

$$\widehat{M_2^1} = \mathrm{Cl}\,\mathrm{Lin}\{\psi_m^k : \ (\mathcal{A}^* - \overline{\lambda_m^k} I)\psi_m^k = 0, \ \lambda_m^k \in \Lambda_1\}, \tag{3.43}$$

two finite-dimensional subspaces of eigenvectors and root-vectors:

$$M_2^2 = \mathrm{Lin}\{\varphi_{i,j} : \ (\mathcal{A} - \lambda_i I)^j \varphi_{i,j} = 0, \ \lambda_i \in \Lambda_2, \ j = 0, \ldots, s_i - 1\},$$

$$\widehat{M_2^2} = \mathrm{Lin}\{\psi_{i,j} : \ (\mathcal{A}^* - \overline{\lambda_i} I)^j \psi_{i,j} = 0, \ \lambda_i \in \Lambda_2, \ j = 0, \ldots, s_i - 1\} \tag{3.44}$$

and the subspace M_2^0, which satisfies

$$M_2 = (\widehat{M_2^1} \oplus \widehat{M_2^2}) \overset{\perp}{\oplus} M_2^0. \tag{3.45}$$

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Thus, we have constructed three $\mathcal{A}$-invariant subspaces: M_2^0, M_2^1 and M_2^2. The main result of this subsection is the following theorem.

Theorem 3.4. *Let $\sigma_1 = \{\mu_1, \ldots, \mu_{\ell_1}\}$ consists of simple eigenvalues only. For any $N \geq N_1$ the decomposition of the spectrum (3.42) and the subspaces given by (3.2), (3.2) and (3.45) define the direct decomposition of the space M_2:*

$$M_2 = M_2^0 \oplus M_2^1 \oplus M_2^2,$$

where the subspaces M_2^0, M_2^1, M_2^2 are $\mathcal{A}$-invariant.

Proof. The proof of this proposition is very similar to the proof of Theorem 3.3. We prove that any element $\xi \in M_2$ allows the representation:

$$\xi = \xi_0 + \sum_{m=1}^{\ell_1} \sum_{|k| \geq N} c_m^k \varphi_m^k + \sum_{i=1}^{r} \sum_{j=0}^{s_i-1} c_{i,j} \varphi_{i,j}, \quad \xi_0 \in M_2^0, \ \sum_{m=1}^{\ell_1} \sum_{|k| \geq N} |c_m^k|^2 < \infty. \tag{3.46}$$

The eigenvectors $\{\varphi_m^k : \lambda_m^k \in \Lambda_1\}$ form a Riesz basis of the closure of their linear span. The finite set of the generalized eigenvectors $\{\varphi_{i,j} : \lambda_i \in \Lambda_2\}$ is also a basis of their linear span. These vectors are linearly independent from eigenvectors $\{\varphi_m^k\}$. Thus eigenvectors $\{\varphi_m^k\} \cup \{\varphi_{i,j}\}$ form a basis of the closure of their linear span. Arguing the same way, we conclude that the set $\{\widehat{\psi}_m^k\} \cup \{\widehat{\psi}_{i,j}\}$ is also a basis of the closure of its linear span, where $\widehat{\psi}_m^k = \frac{1}{\lambda_m^k} \psi_m^k$ and $\widehat{\psi}_{i,j} = \frac{1}{\lambda_i} \psi_{i,j}$. Moreover, without loss of generality, we may assume that the sets $\{\varphi_m^k\} \cup \{\varphi_{i,j}\}$ and $\{\widehat{\psi}_m^k\} \cup \{\widehat{\psi}_{i,j}\}$ are biorthogonal after the normalization.

Using the representations (2.18) and (2.19), we choose eigenvectors with $\|x_m^k\|_{\mathbb{C}^n} = 1$ and $\|y_m^k\|_{\mathbb{C}^n} = 1$. Due to Lemma 3.5 there exists $C > 0$ such that $\|\varphi_m^k\|_{M^2} \leq C$ and $\|\widehat{\psi}_m^k\|_{M^2} \leq C$ for all $m = 1, \ldots, \ell_1$, $|k| \geq N$.

Applying the decomposition (3.33) to vectors φ_m^k, we obtain

$$\varphi_m^k = \gamma_m^k + \sum_{i=1}^{\ell_1} \sum_{|j| \geq N} a_m^k \widehat{\psi}_m^k + \sum_{i=1}^{r} \sum_{j=0}^{s_i-1} a_{i,j} \widehat{\psi}_{i,j}, \quad \gamma_m^k \in M_2^0.$$

Since the sets $\{\varphi\}$ and $\{\psi\}$ are biorthogonal, the last representation can be rewritten as follows:

$$\varphi_m^k = \gamma_m^k + a_m^k \widehat{\psi}_m^k, \quad \gamma_m^k \in M_2^0, \tag{3.47}$$

and, moreover, due to Lemma 2.6 we have

$$a_m^k = \frac{\langle \varphi_m^k, \widehat{\psi}_m^k \rangle_{M_2}}{\|\widehat{\psi}_m^k\|_{M_2}^2} = \frac{\frac{1}{\lambda_m^k} \langle \varphi_m^k, \psi_m^k \rangle_{M_2}}{\|\widehat{\psi}_m^k\|_{M_2}^2} = \frac{-\frac{1}{\lambda_m^k} \langle \Delta'(\lambda_m^k) x_m^k, y_m^k \rangle_{\mathbb{C}^n}}{\|\widehat{\psi}_m^k\|_{M_2}^2}. \tag{3.48}$$

Besides, arguing the same we obtain:

$$\varphi_{i,j} = \gamma_{i,j} + a_{i,j} \widehat{\psi}_{i,j_1}, \quad \gamma_{i,j} \in M_2^0, j_1 = 0, \ldots, s_i - 1.$$

From (3.47) and (3.48) it also follows that

$$\|\gamma_m^k\| \leq \|\varphi_m^k\| + |a_m^k| \|\widehat{\psi}_m^k\| \leq C + \sqrt{|\frac{1}{\lambda} \langle \Delta'(\lambda_m^k) x_m^k, y_m^k \rangle|}. \tag{3.49}$$

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Using the decomposition (3.33) and the relation (3.47), we represent each vector $\xi \in M_2$ as follows:

$$\xi = \widehat{\xi}_0 + \sum_{m=1}^{\ell_1} \sum_{|k| \geq N} b_m^k \widehat{\psi}_m^k + \sum_{i=1}^{r} \sum_{j=0}^{s_i-1} b_{i,j} \widehat{\psi}_{i,j}$$

$$= \widehat{\xi}_0 - \sum_{m=1}^{\ell_1} \sum_{|k| \geq N} \frac{b_m^k}{a_m^k} \gamma_m^k - \sum_{i=1}^{r} \sum_{j=0}^{s_i-1} \frac{b_{i,j}}{a_{i,j}} \gamma_{i,j} + \sum_{m=1}^{\ell_1} \sum_{|k| \geq N} \frac{b_\lambda}{a_m^k} \varphi_m^k + \sum_{i=1}^{r} \sum_{j=0}^{s_i-1} \frac{b_{i,j}}{a_{i,j}} \varphi_{i,j}$$

$$= \xi_0 + \sum_{m=1}^{\ell_1} \sum_{|k| \geq N} c_m^k \varphi_m^k + \sum_{i=1}^{r} \sum_{j=0}^{s_i-1} c_{i,j} \varphi_{i,j}, \qquad (3.50)$$

where $\widehat{\xi}_0 \in M_2^0$, $\sum_{m=1}^{\ell_1} \sum_{|k| \geq N} |b_m^k|^2 < \infty$, $\xi_0 = \widehat{\xi}_0 - \sum_{m=1}^{\ell_1} \sum_{|k| \geq N} \frac{b_m^k}{a_m^k} \gamma_m^k - \sum_{i=1}^{r} \sum_{j=0}^{s_i-1} \frac{b_{i,j}}{a_{i,j}} \gamma_{i,j} \in M_2^0$, $c_m^k = \frac{b_m^k}{a_m^k}$.

To prove the validity of the decomposition (3.2) it is enough to show that $\left| \frac{1}{a_m^k} \right| \leq C_1$ and $\|\gamma_m^k\| \leq C_2$. Taking into account (3.48) and (3.49), the last means to give the estimate

$$0 < C_1 \leq \left| \frac{1}{\lambda_m^k} \langle \Delta'(\lambda_m^k) x_m^k, y_m^k \rangle \right| \leq C_2, \qquad \lambda_m^k \in \Lambda_1, \qquad (3.51)$$

This estimate is proved by Lemma 3.11. Therefore, the representation (3.2) holds for any $\xi \in M_2$, what completes the proof of the theorem. $\qquad \square$

3.3 Auxiliary results

In this subsection we prove several estimates which have been used in the proofs of Theorem 3.3 and Theorem 3.4.

Lemma 3.5. *Let us consider eigenvectors $\varphi_m^k = \varphi(\lambda_m^k)$ and $\psi_m^k = \psi(\overline{\lambda_m^k})$ and their representations (2.18) and (2.19). Let us assume that $\|x_m^k\|_{\mathbb{C}^n} = \|y_m^k\|_{\mathbb{C}^n} = 1$ in these representations. Then for any $N \in \mathbb{N}$ there exists a constant $C > 0$ such that*

$$\|\varphi_m^k\| \leq C, \qquad \frac{1}{|\lambda_m^k|} \|\psi_m^k\| \leq C, \qquad m = 1, \ldots, \ell_1, \ |k| \geq N. \qquad (3.52)$$

In other words the two families of eigenvectors $\{\varphi_m^k : m = 1, \ldots, \ell_1, |k| \geq N\}$ with $\|x_m^k\|_{\mathbb{C}^n} = 1$ and $\left\{ \frac{1}{\lambda_m^k} \psi_m^k : m = 1, \ldots, \ell_1, |k| \geq N \right\}$ with $\|y_m^k\|_{\mathbb{C}^n} = 1$ are bounded.

Proof. Using (2.18) and the relation $\|x_m^k\|_{\mathbb{C}^n} = 1$ we obtain:

$$\begin{aligned}
\|\varphi_m^k\|^2 &= \|(I - e^{-\lambda_m^k A_{-1}}) x_m^k\|^2 + \int_{-1}^{0} \|e^{\lambda_m^k \theta} x_m^k\|^2 \, d\theta \\
&\leq \|I - e^{-\lambda_m^k A_{-1}}\|^2 + \int_{-1}^{0} e^{2\mathrm{Re}\lambda_m^k \theta} \, d\theta \\
&\leq 1 + e^{2\mathrm{Re}\lambda_m^k} \|A_{-1}\|^2 + \frac{1 - e^{-2\mathrm{Re}\lambda_m^k}}{2\mathrm{Re}\lambda_m^k} \leq 1 + \|A_{-1}\|^2 + \frac{1 - e^{-2r}}{2r} \leq C,
\end{aligned}$$

where $r = \max_{k \in \mathbb{N}} r^{(k)}$ and $r^{(k)}$ are the radii of the circles $L_m^k(r^{(k)})$. The last inequality holds since the real function $\frac{1 - e^{-y}}{y}$ decreases monotone from $+\infty$ to 1 when the variable y runs from ∞ to -0.

From (2.19) and since $\|y_m^k\|_{\mathbb{C}^n} = 1$, we have:

$$
\begin{aligned}
\left\|\tfrac{1}{\lambda_m^k}\psi_m^k\right\|^2 &= \tfrac{1}{|\lambda_m^k|^2}\|y_m^k\|^2 + \int_{-1}^{0}\Big\|(\mathrm{e}^{-\overline{\lambda_m^k}\theta} - \tfrac{1}{\lambda_m^k}A_2^*(\theta)+ \\
&\quad + \tfrac{1}{\lambda_m^k}\mathrm{e}^{-\overline{\lambda_m^k}\theta}\int_0^\theta \mathrm{e}^{\overline{\lambda_m^k}s}A_3^*(s)\,\mathrm{d}s + \mathrm{e}^{-\overline{\lambda_m^k}\theta}\int_0^\theta \mathrm{e}^{\overline{\lambda_m^k}s}A_2^*(s)\,\mathrm{d}s)y_m^k\Big\|^2\,\mathrm{d}\theta \\
&\leq \|y_m^k\|^2\left(\tfrac{1}{|\lambda_m^k|^2} + \tfrac{\mathrm{e}^{2\mathrm{Re}\lambda_m^k}-1}{2\mathrm{Re}\lambda_m^k} + \tfrac{1}{|\lambda_m^k|^2}\|A_2^*(\theta)\|_{L_2}^2\right) \\
&\quad + \int_{-1}^0 \mathrm{e}^{-2\mathrm{Re}\lambda_m^k\theta}\,\mathrm{d}\theta\int_{-1}^0 \mathrm{e}^{2\mathrm{Re}\lambda_m^k s}\left(\tfrac{1}{|\lambda_m^k|^2}\|A_3^*(s)\| + \|A_2^*(s)\|\right)\,\mathrm{d}s \\
&\leq \tfrac{1}{|\lambda_m^k|^2} + \tfrac{1}{|\lambda_m^k|^2}\|A_2^*(\theta)\|_{L_2}^2 + \tfrac{\mathrm{e}^{2\mathrm{Re}\lambda_m^k}-1}{2\mathrm{Re}\lambda_m^k}\left(1 + \tfrac{1}{|\lambda_m^k|^2}\|A_3^*(\theta)\|_{L_2}^2 + \|A_2^*(\theta)\|_{L_2}^2\right) \\
&\leq \left(\tfrac{1}{|\lambda_m^k|^2} + 1\right)\left(\|A_2^*(\theta)\|_{L_2}^2 + 1\right) + \tfrac{1}{|\lambda_m^k|^2}\|A_3^*(\theta)\|_{L_2}^2 \leq C,
\end{aligned}
$$

where $\|A_i^*(\theta)\|_{L_2} = \|A_i^*(\theta)\|_{L_2(-1,0;\mathbb{C}^{n\times n})}$. Here we used the fact that the real function $\frac{\mathrm{e}^y-1}{y}$ increases monotone from 0 to 1 when the variable y runs from ∞ to -0. $\square$

Remark 3.6. *We notice that the norm of eigenvectors ψ_m^k (assuming $\|y_m^k\| = 1$) increases infinitely when $k \to \infty$. This could be seen on the example of eigenvectors $\widetilde{\psi}_m^k$ of the operator $\widetilde{A}^*$ ($A_2^*(\theta) = A_3^*(\theta) \equiv 0$):*

$$
\|\widetilde{\psi}_m^k\|^2 = \|y_m^k\|^2 + \int_{-1}^0 \|\overline{\lambda_m^k}\,\mathrm{e}^{-\overline{\lambda_m^k}\theta}y_m^k\|^2\,\mathrm{d}\theta = \|y_m^k\|^2\left(1 + |\lambda_m^k|^2\frac{\mathrm{e}^{2\mathrm{Re}\lambda_m^k}-1}{2\mathrm{Re}\lambda_m^k}\right)
$$

$$
\geq (1 + C|\lambda_m^k|^2) \to +\infty, \qquad k \to \infty.
$$

To formulate the next proposition we introduce the matrices

$$
R_m = \begin{pmatrix} \widehat{R}_m & 0 \\ 0 & I \end{pmatrix}, \qquad \widehat{R}_m = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \mathbb{C}^{m\times m}, \qquad m = 1, \ldots, \ell_1,
$$

where $I = I_{n-m}$ is the is the identity matrix of the dimension $n - m$. Obviously, $R_1 = I$ and $R_m^{-1} = R_m^* = R_m$ for all $m = 1, \ldots, \ell_1$.

Lemma 3.7. *Assume that $\sigma_1 = \{\mu_1, \ldots, \mu_{\ell_1}\}$ consists of simple eigenvalues only. There exists $N \in \mathbb{N}$ such that for any $\lambda_m^k \in \Lambda_1$, $|k| \geq N$ and the corresponding matrix $\Delta(\lambda_m^k)$ there exist matrices $P_{m,k}$, $Q_{m,k}$ of the form*

$$
P_{m,k} = \begin{pmatrix} 1 & -p_2 & \cdots & -p_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \qquad Q_{m,k} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -q_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -q_n & 0 & \cdots & 1 \end{pmatrix} \tag{3.53}
$$

such that the product $\frac{1}{\lambda_m^k}P_{m,k}R_m\Delta(\lambda_m^k)R_mQ_{m,k}$ has the following form:

$$
\frac{1}{\lambda_m^k}P_{m,k}R_m\Delta(\lambda_m^k)R_mQ_{m,k} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & S_{m,k} & \\ 0 & & & \end{pmatrix}, \qquad \det S_{m,k} \neq 0, \tag{3.54}
$$

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Moreover, for any $\varepsilon > 0$ there exists $N \in \mathbb{Z}$ such that for any $|k| \geq N$ the components $p_i = p_i(m, k)$, $q_i = q_i(m, k)$ of the matrices (3.53) may be estimated as follows:

$$|p_i| \leq \varepsilon, \quad |q_i| \leq \varepsilon, \quad i = 2, \ldots, n. \tag{3.55}$$

Proof. We begin with the analysis of the structure of the matrix

$$\frac{1}{\lambda_m^k} R_m \Delta(\lambda_m^k) R_m = -I + e^{-\lambda_m^k} R_m A_{-1} R_m + \int_{-1}^0 e^{\lambda_m^k s} R_m \left(A_2(s) + \frac{1}{\lambda_m^k} A_3(s) \right) R_m \, ds.$$

Since the matrix A_{-1} is in Jordan form (2.16), then the multiplication of A_{-1} on R_m from the left and from the right changes the places of the one-dimensional Jordan blocks μ_1 and μ_m:

$$R_m A_{-1} R_m = \begin{pmatrix} \mu_m & 0 \\ 0 & S \end{pmatrix}, \qquad S \in C^{(n-1) \times (n-1)}.$$

We introduce the notation

$$\int_{-1}^0 e^{\lambda s} R_m \left(A_2(s) + \frac{1}{\lambda} A_3(s) \right) R_m \, ds = \begin{pmatrix} \varepsilon_{11}(\lambda) & \cdots & \varepsilon_{1n}(\lambda) \\ \vdots & \ddots & \vdots \\ \varepsilon_{n1}(\lambda) & \cdots & \varepsilon_{nn}(\lambda) \end{pmatrix}. \tag{3.56}$$

According to Proposition 4.6, elements of the matrix (3.56) tends to zero when $|\mathrm{Im}\lambda| \to \infty$ (and $|\mathrm{Re}\lambda| \leq C < \infty$). Thus, $|\varepsilon_{ij}(\lambda_m^k)| \to 0$ when $k \to \infty$.

In the introduced notation the singular matrix $\frac{1}{\lambda_m^k} R_m \Delta(\lambda_m^k) R_m$ has the following form:

$$\frac{1}{\lambda_m^k} R_m \Delta(\lambda_m^k) R_m = \begin{pmatrix} -1 + e^{-\lambda_m^k} \mu_m + \varepsilon_{11}(\lambda_m^k) & \varepsilon_{12}(\lambda_m^k) & \cdots & \varepsilon_{1n}(\lambda_m^k) \\ \varepsilon_{21}(\lambda_m^k) & & & \\ \vdots & & S_{m,k} & \\ \varepsilon_{n1}(\lambda_m^k) & & & \end{pmatrix}, \tag{3.57}$$

where

$$S_{m,k} = \left(-I_{n-1} + e^{-\lambda_m^k} S + \begin{pmatrix} \varepsilon_{22}(\lambda_m^k) & \cdots & \varepsilon_{2n}(\lambda_m^k) \\ \vdots & \ddots & \vdots \\ \varepsilon_{n2}(\lambda_m^k) & \cdots & \varepsilon_{nn}(\lambda_m^k) \end{pmatrix} \right) \tag{3.58}$$

and I_{n-1} is the identity matrix of the dimension $n - 1$. Let us prove that

$$\det S_{m,k} \neq 0. \tag{3.59}$$

Consider the identity $-I_{n-1} + e^{-\lambda_m^k} S = -I_{n-1} + e^{-\widetilde{\lambda}_m^k} S + (e^{-\lambda_m^k} - e^{-\widetilde{\lambda}_m^k}) S$, where $\widetilde{\lambda}_m^k = i(\arg \mu_m + 2\pi k)$ is an eigenvalue of the operator $\widetilde{\mathcal{A}}$. Since $e^{\widetilde{\lambda}_m^k} = \mu_m$ we have

$$-I + e^{-\widetilde{\lambda}_m^k} R_m A_{-1} R_m = \begin{pmatrix} 0 & 0 \\ 0 & -I_{n-1} + e^{-\widetilde{\lambda}_m^k} S \end{pmatrix},$$

and since the multiplicity of the eigenvalue $\mu_m \in \sigma_1$ equals 1, we conclude that $\det(-I_{n-1} + e^{-\widetilde{\lambda}_m^k} S) \neq 0$. Since $|\lambda_m^k - \widetilde{\lambda}_m^k| \to 0$ when $k \to \infty$, then for any $\varepsilon > 0$ there exists $N > 0$ such that for $k : |k| \geq N$ the estimates $|e^{-\lambda_m^k} - e^{-\widetilde{\lambda}_m^k}| \|S\| \leq \frac{\varepsilon}{2}$ and $|\varepsilon_{ij}(\lambda_m^k)| \leq \frac{\varepsilon}{2}$ hold. Thus, we have that $\det S_{m,k} = \det(-I_{n-1} + e^{-\widetilde{\lambda}_m^k} S + B_{m,k})$, where the absolute value of each component of $B_{m,k}$ is less than ε. Therefore, there exists $N > 0$ such that $S_{m,k}$ is invertible and we obtain the relation (3.59).

Since $\det S_{m,k} \neq 0$ then from the relation (3.57) we conclude that the first row of the matrix $\frac{1}{\lambda_m^k} R_m \Delta(\lambda_m^k) R_m$ is a linear combination of all other rows and the first column is a linear combination of all other columns:

$$\varepsilon_{1i}(\lambda_m^k) = p_2 s_{2i} + \ldots + p_n s_{ni}, \quad i = 2, \ldots, n$$

$$\varepsilon_{j1}(\lambda_m^k) = q_2 s_{j2} + \ldots + q_n s_{jn}, \quad j = 2, \ldots, n. \tag{3.60}$$

where $s_{ij} = s_{ij}(m,k)$, $2 \leq i, j \leq n$ are the components of the matrix $S_{m,k}$.

Let us consider the matrices $P_{m,k}$, $Q_{m,k}$ of the form (3.53) with the coefficients $p_2, \ldots, p_n$ and $q_2, \ldots, q_n$ defined by (3.3). Direct computations gives us that $\frac{1}{\lambda_m^k} P_{m,k} R_m \Delta(\lambda_m^k) R_m Q_{m,k}$ is of the form (3.54), i.e.:

$$\frac{1}{\lambda_m^k} P_{m,k} R_m \Delta(\lambda_m^k) R_m Q_{m,k} = \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 0 & & & \\ \vdots & & S_{m,k} & \\ 0 & & & \end{pmatrix}.$$

Let us estimate the coefficients $p_2, \ldots, p_n$ and $q_2, \ldots, q_n$. The equations (3.3) may be rewritten in the form:

$$v_1 = (S_{m,k})^T w_1, \qquad v_2 = S_{m,k} w_2,$$

where $v_1 = (\varepsilon_{12}(\lambda_m^k), \ldots, \varepsilon_{1n}(\lambda_m^k))^T$, $w_1 = (p_2, \ldots, p_n)^T$, $v_2 = (\varepsilon_{21}(\lambda_m^k), \ldots, \varepsilon_{n1}(\lambda_m^k))^T$, $w_2 = (q_2, \ldots, q_n)^T$. Since $\det S_{m,k} \neq 0$ then

$$w_1 = (S_{m,k}^{-1})^T v_1, \qquad w_2 = S_{m,k}^{-1} v_2$$

and since the values $\varepsilon_{ij}(\lambda_m^k)$ are small then to show (3.55) we have to prove that the estimate $\|S_{m,k}^{-1}\| \leq C$, $C > 0$ holds for all $k : |k| \geq N$.

As we have shown above $S_{m,k} = -I_{n-1} + \frac{1}{\mu_m} S + B_{m,k}$, where elements of the matrices $B_{m,k}$ tend to zero when $k \to \infty$. Thus, there exists $N \in \mathbb{Z}$ such that for all $k : |k| \geq N$ the norm of the matrix $\widetilde{B}_{m,k} \stackrel{\text{def}}{=} - \left(-I_{n-1} + \frac{1}{\mu_m} S \right)^{-1} B_{m,k}$ is small enough, say $\|\widetilde{B}_{m,k}\| < \frac{1}{2}$. Thus, there exists the inverse matrix of $I_{n-1} - \widetilde{B}_{m,k}$ for every $|k| \geq N$, and these inverse matrices are bounded uniformly by k:

$$\|(I_{n-1} - \widetilde{B}_{m,k})^{-1}\| = \| \sum_{i=0}^{\infty} (\widetilde{B}_{m,k})^i \| \leq C_1, \quad |k| \geq N.$$

Thus, we obtain the estimate

$$\|S_{m,k}^{-1}\| = \left\| (I_{n-1} - \widetilde{B}_{m,k})^{-1} \left(-I_{n-1} + \frac{1}{\mu_m} S \right)^{-1} \right\| \leq C_1 \left\| \left(-I_{n-1} + \frac{1}{\mu_m} S \right)^{-1} \right\| \leq C,$$

what completes the proof of the lemma. $\qquad \square$

Corollary 3.8. *The matrix function* $\widehat{\Delta}_{m,k}(\lambda) \stackrel{\text{def}}{=} \frac{1}{\lambda} P_{m,k} R_m \Delta(\lambda) R_m Q_{m,k}$, *where* $P_{m,k}$, $Q_{m,k}$ *are given by (3.53), allows the following representation in a neighborhood* $U(\lambda_m^k)$ *of the corresponding eigenvalue* $\lambda_m^k \in \Lambda_1$, $|k| \geq N$:

$$\widehat{\Delta}_{m,k}(\lambda) = \begin{pmatrix} (\lambda - \lambda_m^k) r_{11}^{m,k}(\lambda) & (\lambda - \lambda_m^k) r_{12}^{m,k}(\lambda) & \ldots & (\lambda - \lambda_m^k) r_{1n}^{m,k}(\lambda) \\ (\lambda - \lambda_m^k) r_{21}^{m,k}(\lambda) & & & \\ \vdots & & S_{m,k}(\lambda) & \\ (\lambda - \lambda_m^k) r_{n1}^{m,k}(\lambda) & & & \end{pmatrix}, \tag{3.61}$$

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where the functions $r_{ij}^{m,k}(\lambda)$ *are analytic in* $U(\lambda_m^k)$. *Moreover,*

$$r_{11}^{m,k}(\lambda_m^k) \neq 0, \qquad |r_{11}^{m,k}(\lambda_m^k)| \to 1, \ k \to \infty. \tag{3.62}$$

Proof. Since $\Delta(\lambda)$ is analytic, then all the components of the matrix function $\widehat{\Delta}_{m,k}(\lambda) = \frac{1}{\lambda} P_{m,k} R_m \Delta(\lambda) R_m Q_{m,k}$ are analytic in a neighborhood of the point λ_m^k. Moreover, since the matrix $\widehat{\Delta}_{m,k}(\lambda_m^k)$ has the form (3.54), then we conclude that $\widehat{\Delta}_{m,k}(\lambda)$ is of the form (3.61).

Let us prove the relation (3.62). If we assume that $r_{11}^{m,k}(\lambda_m^k) = 0$, then $(\lambda - \lambda_m^k)r_{11}^{m,k}(\lambda) = (\lambda - \lambda_m^k)^2 \widehat{r}_{11}^{m,k}(\lambda)$, where $\widehat{r}_{11}^{m,k}(\lambda)$ is analytic. The last implies that the multiplicity of the root $\lambda = \lambda_m^k$ of the equation $\det \widehat{\Delta}_{m,k}(\lambda) = 0$ is greater or equal to 2, i.e. $\det \widehat{\Delta}_{m,k}(\lambda) = (\lambda - \lambda_m^k)^2 r(\lambda)$, where $r(\lambda)$ is an analytic function. Indeed, decomposing $\det \widehat{\Delta}_{m,k}(\lambda)$ by the elements of the first row, we see that all the term of this decomposition have the common multiplier $(\lambda - \lambda_m^k)^2$. Thus, we obtain that the multiplicity of $\lambda = \lambda_m^k$ as the root of the equation $\det \Delta(\lambda) = 0$ is greater or equal to 2, what contradicts to the assumption that λ_m^k is an eigenvalue of the multiplicity one of the operator $\mathcal{A}$.

Taking into account (3.57) and the form of the transformations (3.54), we see that

$$(\lambda - \lambda_m^k)r_{11}^{m,k}(\lambda) = \left(-1 + e^{-\lambda}\mu_m + \varepsilon_{11}(\lambda)\right) - \sum_{i=2}^n p_i \varepsilon_{i1}(\lambda) - \sum_{j=2}^n q_j \varepsilon_{1j}(\lambda). \tag{3.63}$$

Differentiating (3.63) by λ and substituting $\lambda = \lambda_m^k$, we obtain

$$r_{11}^{m,k}(\lambda_m^k) = -e^{-\lambda_m^k}\mu_m + \left(\varepsilon_{11}(\lambda) - \sum_{i=2}^n p_i \varepsilon_{i1}(\lambda) - \sum_{j=2}^n q_j \varepsilon_{1j}(\lambda)\right)'_{\lambda = \lambda_m^k}.$$

The terms $(\varepsilon_{ij}(\lambda))'$ are of the form

$$(\varepsilon_{ij}(\lambda))' = \int_{-1}^0 e^{\lambda s}\left(sA_2(s) + \frac{s}{\lambda}A_3(s) - \frac{1}{\lambda^2}A_3(s)\right)_{ij} ds,$$

therefore, due to Proposition 4.6 and Lemma 3.7, we conclude that

$$\left(\varepsilon_{11}(\lambda) - \sum_{i=2}^n p_i \varepsilon_{i1}(\lambda) - \sum_{j=2}^n q_j \varepsilon_{1j}(\lambda)\right)'_{\lambda = \lambda_m^k} \to 0, \quad k \to \infty.$$

Since $-e^{-\lambda_m^k}\mu_m \to -1$ when $k \to \infty$, then we obtain the relation (3.62) and, in particular, there exists a constant $C > 0$ and an integer N such that for $|k| > N$ we have

$$0 < C \leq \left|r_{11}^{m,k}(\lambda_m^k)\right|. \tag{3.64}$$

The last completes the proof of the proposition. □

Remark 3.9. *The same arguments gives us that*

$$|r_{i1}^{m,k}(\lambda_m^k)| \to 0, \ |r_{1j}^{m,k}(\lambda_m^k)| \to 0, \quad k \to \infty. \tag{3.65}$$

for all $i = 2, \ldots, n$ *and for all* $i = 2, \ldots, n$.

Proof. Indeed, let us consider $r_{1j}^{m,k}(\lambda)$ for $j = 2, \ldots, n$ and use the fact that A_{-1} is in a Jordan form:

$$(\lambda - \lambda_m^k)r_{1j}^{m,k}(\lambda) = \varepsilon_{1j}(\lambda) - \sum_{i=2}^{n} p_i\varepsilon_{ij}(\lambda) + p_j\left(-1 + e^{-\lambda}\mu\right) + p_{j-1}c,$$

where $\mu \in \sigma(A_{-1})$ and the constant $c = 0$ if μ is geometrically simple or, otherwise, $c = 1$. Thus, we obtain

$$r_{1j}^{m,k}(\lambda_m^k) = -p_j e^{-\lambda_m^k}\mu + \left(\varepsilon_{1j}(\lambda) - \sum_{i=2}^{n} p_i\varepsilon_{i1}(\lambda)\right)'_{\lambda=\lambda_m^k}$$

and since $p_i = p_i(m,k) \to 0$ when $k \to \infty$ due to Lemma 3.7, then we conclude that $|r_{1j}^{m,k}(\lambda_m^k)| \to 0$ when $k \to \infty$. $\qquad\square$

Remark 3.10. *Direct computations give us:*

$$P_{m,k}^{-1} = \begin{pmatrix} 1 & p_2 & \cdots & p_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad Q_{m,k}^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ q_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q_n & 0 & \cdots & 1 \end{pmatrix}. \tag{3.66}$$

Lemma 3.11. *Let $\sigma_1 = \{\mu_1, \ldots, \mu_{\ell_1}\}$ consists of simple eigenvalues only. There exist constants $0 < C_1 < C_2$, $N \in \mathbb{Z}$ such that for any $\lambda_m^k \in \Lambda_1$, $|k| \geq N$ the following estimate holds:*

$$0 < C_1 \leq \left|\frac{1}{\lambda_m^k}\langle\Delta'(\lambda_m^k)x_m^k, y_m^k\rangle\right| \leq C_2, \tag{3.67}$$

where $\Delta'(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta(\lambda)$; $x_m^k = x(\lambda_m^k)$, $y_m^k = y(\overline{\lambda}_m^k)$ are defined by (2.18), (2.19) and $\|x_m^k\| = \|y_m^k\| = 1$.

Proof. First, we prove the estimate (3.67) for eigenvalues $\lambda_1^k \in \Lambda_1$. Since $x_1^k \in \mathrm{Ker}\Delta(\lambda_1^k)$, then

$$0 = \frac{1}{\lambda_1^k}P_{1,k}^{-1}P_{1,k}\Delta(\lambda_1^k)Q_{1,k}Q_{1,k}^{-1}x_1^k = P_{1,k}^{-1}\widehat{\Delta}_{1,k}(\lambda_1^k)Q_{1,k}^{-1}x_1^k, \tag{3.68}$$

where $P_{1,k}$, $Q_{1,k}$, $\widehat{\Delta}_{1,k}(\lambda_1^k)$ are defined by (3.53), (3.54). Thus, $Q_{1,k}^{-1}x_1^k \in \mathrm{Ker}\widehat{\Delta}_{1,k}(\lambda_1^k)$ and, taking into account the form (3.54) of the matrix $\widehat{\Delta}_{1,k}(\lambda_1^k)$, we conclude that $Q_{1,k}^{-1}x_1^k = (\widehat{x}_1, 0, \ldots, 0)^T$, $\widehat{x}_1 \neq 0$. On the other hand, multiplying directly $Q_{1,k}^{-1}$ given by (3.66) on $x_1^k = ((x_1^k)_1, \ldots, (x_1^k)_n)^T$, we conclude that $\widehat{x}_1 = (x_1^k)_1$ and $(x_1^k)_i = -q_i(x_1^k)_1$, $i = 2, \ldots, n$.

Due to the relation (3.55), for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $k : |k| \geq N$ we have:

$$1 = \|x_1^k\|^2 = |(x_1^k)_1|^2\left(1 + |q_2|^2 + \ldots + |q_n|^2\right) \leq |(x_1^k)_1|^2\left(1 + (n-1)\varepsilon^2\right)$$

what implies that $|(x_1^k)_1| \to 1$ when $k \to \infty$. Finally, we obtain

$$Q_{1,k}^{-1}x_1^k = ((x_1^k)_1, 0, \ldots, 0)^T, \qquad 0 < C \leq |(x_1^k)_1| \leq 1, \ |k| \geq N. \tag{3.69}$$

Conjugating (3.54), we have the relation

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & S_{1,k}^* & \\ 0 & & & \end{pmatrix} = \left(\frac{1}{\lambda_1^k}P_{1,k}\Delta(\lambda_1^k)Q_{1,k}\right)^* = \frac{1}{\overline{\lambda}_1^k}Q_{1,k}^*\Delta^*(\overline{\lambda}_1^k)P_{1,k}^* = \widehat{\Delta}_{1,k}^*(\overline{\lambda}_1^k). \tag{3.70}$$

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Let us use the fact that $y_1^k \in \mathrm{Ker}\Delta^*(\overline{\lambda_1^k})$:

$$0 = \frac{1}{\overline{\lambda_1^k}}(Q_{1,k}^*)^{-1}Q_{1,k}^*\Delta^*(\overline{\lambda_1^k})P_{1,k}^*(P_{1,k}^*)^{-1}y_1^k = (Q_{1,k}^*)^{-1}\widehat{\Delta}_{1,k}^*(\overline{\lambda_1^k})(P_{1,k}^*)^{-1}y_1^k. \qquad (3.71)$$

From the last we obtain that $(P_{1,k}^*)^{-1}y_1^k \in \mathrm{Ker}\widehat{\Delta}_{1,k}^*(\overline{\lambda_1^k})$, and, taking into account the left-hand side of (3.70), we conclude that $(P_{1,k}^*)^{-1}y_1^k = (\widehat{y}_1, 0, \ldots, 0)^T$, $\widehat{y}_1 \neq 0$. Multiplying $(P_{1,k}^*)^{-1}$ on $y_1^k = ((y_1^k)_1, \ldots, (y_1^k)_1)^T$ we obtain the relations $\widehat{y}_1 = (y_1^k)_1$ and $(y_1^k)_i = -\overline{p}_i(y_1^k)_1$, $i = 2, \ldots, n$. Thus, due to (3.55), any $\varepsilon > 0$ and $k : |k| \geq N$ we have:

$$1 = \|y_1^k\|^2 = |(y_1^k)_1|^2\left(1 + |\overline{p}_2|^2 + \ldots + |\overline{p}_n|^2\right) \leq |(y_1^k)_1|^2\left(1 + (n-1)\varepsilon^2\right)$$

and we conclude that $|(y_1^k)_1| \to 1$ when $k \to \infty$. Finally,

$$(P_{1,k}^*)^{-1}y_1^k = ((y_1^k)_1, 0, \ldots, 0)^T, \qquad 0 < C \leq |(y_1^k)_1| \leq 1, \ |k| \geq N. \qquad (3.72)$$

Differentiating (3.61) by λ and putting $\lambda = \lambda_1^k$, we obtain

$$\begin{pmatrix} r_{11}^{1,k}(\lambda_1^k) & r_{12}^{1,k}(\lambda_1^k) & \cdots & r_{1n}^{1,k}(\lambda_1^k) \\ r_{21}^{1,k}(\lambda_1^k) & & & \\ \vdots & & S'_{m,k}(\lambda_1^k) & \\ r_{n1}^{1,k}(\lambda_1^k) & & & \end{pmatrix} = \widehat{\Delta}'_{1,k}(\lambda_1^k) = \left(\frac{1}{\lambda}P_{1,k}\Delta(\lambda)Q_{1,k}\right)'_{\lambda=\lambda_1^k}$$

$$= P_{1,k}\left(\frac{1}{\lambda_1^k}\Delta'(\lambda_1^k) - \frac{1}{(\lambda_1^k)^2}\Delta(\lambda_1^k)\right)Q_{1,k}. \qquad (3.73)$$

Using (3.3) and the relation $x_1^k \in \mathrm{Ker}\Delta(\lambda_1^k)$, we obtain

$$\begin{aligned} \frac{1}{\lambda_1^k}\left\langle \Delta'(\lambda_1^k)x_1^k, \ y_1^k \right\rangle &= \left\langle P_{1,k}^{-1}P_{1,k}\left(\frac{1}{\lambda_1^k}\Delta'(\lambda_1^k) - \frac{1}{(\lambda_1^k)^2}\Delta(\lambda_1^k)\right)Q_{1,k}Q_{1,k}^{-1}x_1^k, \ y_1^k \right\rangle \\ &= \left\langle P_{1,k}\left(\frac{1}{\lambda_1^k}\Delta'(\lambda_1^k) - \frac{1}{(\lambda_1^k)^2}\Delta(\lambda_1^k)\right)Q_{1,k}Q_{1,k}^{-1}x_1^k, \ (P_{1,k}^{-1})^*y_1^k \right\rangle \\ &= \left\langle \widehat{\Delta}'_{1,k}(\lambda_1^k)Q_{1,k}^{-1}x_1^k, \ (P_{1,k}^{-1})^*y_1^k \right\rangle. \end{aligned} \qquad (3.74)$$

Finally, using the representation (3.3) of the matrix $\widehat{\Delta}'_{1,k}(\lambda_1^k)$ and representations (3.69), (3.72) of the vectors $Q_{1,k}^{-1}x_1^k$, $(P_{1,k}^{-1})^*y_1^k$, we conclude that

$$\frac{1}{\lambda_1^k}\left\langle \Delta'(\lambda_1^k)x_1^k, \ y_1^k \right\rangle = r_{11}^{1,k}(\lambda_1^k)(x_1^k)_1(y_1^k)_1. \qquad (3.75)$$

Moreover, taking into account the estimate (3.62) of Corollary 3.8 and (3.69), (3.72), we obtain the estimate (3.67), what proves the lemma for the case of eigenvalues λ_1^k, i.e. for $m = 1$.

Let us now prove the estimate (3.67) for $\lambda_m^k \in \Lambda_1$, $m = 2, \ldots, \ell_1$. In this case the idea of the proof remains the same but the arguing appears to be more cumbersome. In the proof we omit some detailed explanations that were given above for the case $m = 1$.

Let us consider the product $R_m\Delta(\lambda_m^k)R_m$. Using the relation $x_m^k \in \mathrm{Ker}\Delta(\lambda_m^k)$, we have

$$0 = \frac{1}{\lambda_m^k}R_mP_{m,k}^{-1}P_{m,k}R_m\Delta(\lambda_m^k)R_mQ_{m,k}Q_{m,k}^{-1}R_mx_m^k = R_mP_{m,k}^{-1}\widehat{\Delta}_{m,k}(\lambda_m^k)Q_{m,k}^{-1}R_mx_m^k. \qquad (3.76)$$

Thus, $Q_{m,k}^{-1}R_m x_m^k \in \mathrm{Ker}\widehat{\Delta}_{m,k}(\lambda_m^k)$ and from the explicit form (3.54) of $\widehat{\Delta}_{m,k}(\lambda_m^k)$ we conclude that $Q_{m,k}^{-1}R_m x_m^k = (\widehat{x}_1, 0, \ldots, 0)^T$, $\widehat{x}_1 \neq 0$. Multiplying $Q_{m,k}^{-1}$ on R_m from the right, we changes places the first and the m-th column of $Q_{m,k}^{-1}$, therefore, we obtain:

$$(x_m^k)_m = \widehat{x}_1, \ (x_m^k)_1 = -q_m(x_m^k)_m, \ (x_m^k)_i = -q_i(x_m^k)_m, \ i = 2, \ldots, n, i \neq m.$$

Thus, taking into account (3.55), for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $k : |k| \geq N$ we have:

$$1 = \|x_m^k\|^2 \leq |(x_m^k)_m|^2(1 + (n-1)\varepsilon^2)$$

and, thus, $|(x_m^k)_m| \to 1$ when $k \to \infty$. Therefore,

$$Q_{m,k}^{-1}R_m x_m^k = \left((x_m^k)_m, 0, \ldots, 0\right)^T, \qquad 0 < C \leq |(x_m^k)_m| \leq 1, \ |k| \geq N. \tag{3.77}$$

The similar arguing gives us that

$$(P_{m,k}^{-1})^* R_m y_m^k = \left((y_m^k)_m, 0, \ldots, 0\right)^T, \qquad 0 < C \leq |(y_m^k)_m| \leq 1, \ |k| \geq N. \tag{3.78}$$

Differentiating (3.61) by λ and putting $\lambda = \lambda_m^k$, we obtain

$$\begin{pmatrix} r_{11}^{m,k}(\lambda_m^k) & r_{12}^{m,k}(\lambda_m^k) & \ldots & r_{1n}^{m,k}(\lambda_m^k) \\ r_{21}^{m,k}(\lambda_m^k) & & & \\ \vdots & & S'_{m,k}(\lambda_m^k) & \\ r_{n1}^{m,k}(\lambda_m^k) & & & \end{pmatrix} = \widehat{\Delta}'_{m,k}(\lambda_m^k)$$

$$= P_{m,k}R_m\left(\frac{1}{\lambda_m^k}\Delta'(\lambda_m^k) - \frac{1}{(\lambda_m^k)^2}\Delta(\lambda_m^k)\right)R_m Q_{m,k}. \tag{3.79}$$

Finally, using (3.77), (3.78), (3.3) and the relation $x_1^k \in \mathrm{Ker}\Delta(\lambda_1^k)$, we obtain

$$\frac{1}{\lambda_m^k}\left\langle \Delta'(\lambda_m^k)x_m^k, \ y_m^k \right\rangle =$$
$$= \left\langle R_m P_{m,k}^{-1}P_{m,k}R_m\left(\frac{1}{\lambda_m^k}\Delta'(\lambda_m^k) - \frac{1}{(\lambda_m^k)^2}\Delta(\lambda_m^k)\right)R_m Q_{m,k}Q_{m,k}^{-1}R_m x_m^k, \ y_m^k \right\rangle$$
$$= \left\langle \widehat{\Delta}'_{m,k}(\lambda_m^k)R_m Q_{m,k}^{-1}x_m^k, \ (P_{m,k}^{-1})^* R_m y_m^k \right\rangle$$
$$= r_{11}^{m,k}(\lambda_m^k)(x_m^k)_m(y_m^k)_m.$$

To complete the proof of the lemma it remains to apply the estimates (3.62), (3.77) and (3.78). $\square$

4 Boundedness of the resolvent on invariant subspaces

In this section we prove the exponential stability of the restriction of the semigroup $\{e^{t\mathcal{A}}\}_{t\geq 0}$ onto M_2^0 (i.e. the semigroup $\{e^{t\mathcal{A}}|_{M_2^0}\}_{t\geq 0}$), where the invariant subspace M_2^0 is defined in Section 3 by (3.33).

To show this we use the following well-known equivalent condition of exponential stability (see e.g. [27, p.119] or [12, p.139]):
Let $T(t)$ be a C_0-semigroup on a Hilbert space H with a generator A. Then $T(t)$ is exponentially stable if and only if the following conditions hold:
  1. $\{\lambda : \mathrm{Re}\lambda \geq 0\} \subset \rho(A)$;
  2. $\|R(\lambda, A)\| \leq M$ for all $\{\lambda : \mathrm{Re}\lambda \geq 0\}$ and for some constant $M > 0$.
Thus, we reformulate our aim as follows.

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Theorem 4.1 (on resolvent boundedness). *Let $\sigma_1 = \{\mu_1, \ldots, \mu_{\ell_1}\}$ consists of simple eigen-values only. On the subspace M_2^0 defined by (3.33) the restriction of the resolvent $R(\lambda, \mathcal{A})|_{M_2^0}$ is uniformly bounded for $\lambda : \operatorname{Re}\lambda \geq 0$, i.e. there exists $C > 0$ such that*

$$\|R(\lambda, \mathcal{A})x\| \leq C\|x\|, \quad x \in M_2^0. \tag{4.80}$$

Let us briefly describe the ideas of the proof. From the explicit form of the resolvent (2.13) we conclude that the main difficulty is to prove the uniform boundedness of the term $\triangle^{-1}(\lambda)D(z, \xi, \lambda)$ in neighborhoods of the eigenvalues of $\mathcal{A}$ located close to the imaginary axis. Indeed, since $\det \triangle(\lambda_m^k) = 0$ for $\lambda_m^k \in \Lambda_1$ and $\operatorname{Re}\lambda_m^k \to 0$ when $k \to \infty$ then norm of $\triangle^{-1}(\lambda)$ grows infinitely when $\operatorname{Re}\lambda \to 0$ and $\operatorname{Im}\lambda \to \infty$ simultaneously. However, the product $\triangle^{-1}(\lambda)D(z, \xi, \lambda)$ turns out to be bounded for $(z, \xi(\cdot)) \in M_2^0$.

Lemma 4.2. *The vector-function $\triangle^{-1}(\lambda)D(z, \xi, \lambda) : M_2^0 \times \mathbb{C}^n \to \mathbb{C}^n$ is uniformly bounded in neighborhoods $U_\delta(\lambda_m^k)$ of eigenvalues $\lambda_m^k \in \Lambda_1$ for some fixed $\delta > 0$, i.e.:*
1.for any $k : |k| > N$ and $m = 1, \ldots, \ell_1$ there exists a constant $C_{m,k}$ such that the estimate $\|\triangle^{-1}(\lambda)D(z, \xi, \lambda)\| \leq C_{m,k}\|(z, \xi(\cdot))\|_{M_2}$ holds for all $\lambda \in U_\delta(\lambda_m^k)$ and $(z, \xi(\cdot)) \in M_2^0$.
2. there exists a constant $C > 0$ such that $C_{m,k} \leq C$ for all $m = 1, \ldots, \ell_1$, $k : |k| > N$.

The proof of this lemma is technically difficult. It essentially uses the following relation.

Lemma 4.3. *For any vector $g = (z, \xi(\cdot))^T \in M_2^0$ and for any eigenvalue $\lambda_m^k \in \Lambda_1$ the following relation holds:*

$$D(z, \xi, \lambda_m^k) \in \operatorname{Im}\triangle(\lambda_m^k). \tag{4.81}$$

The complete proofs of the mentioned propositions are given in the next subsection.

Remark 4.4. *Theorem 4.1 holds also for the subspace M_2^0 defined by (3.45) under the assumption $\Lambda_2 = \emptyset$. The proof remains the same.*

4.1 The proof of Theorem 4.1

We begin with several auxiliary propositions.

Proposition 4.5. *If the vector $y \in \operatorname{Im}A$, $A \in \mathbb{C}^{n \times n}$, then for any two matrices P, Q such that $\det Q \neq 0$ the relation $Py \in \operatorname{Im}(PAQ)$ holds.*

Proof. The relation $y \in \operatorname{Im}A$ means that there exists a vector x such that $Ax = y$. Since Q is non-singular then there exists a vector x_1 such that $x = Qx_1$. Therefore, $AQx_1 = y$ and, multiplying on P from the left we obtain $PAQx_1 = Py$. $\square$

Proposition 4.6. *Let $L_0 \subset \mathbb{C}$ be a bounded closed set and $f(s) \in L_2[-1, 0]$. Denote by $a_k(\lambda) = \int\limits_{-1}^{0} e^{2\pi i k s} e^{\lambda s} f(s)\, ds$, $\lambda \in L_0$, $k \in \mathbb{Z}$. Then $|a_k(\lambda)| \to 0$ when $k \to \infty$ uniformly on the set L_0.*

Proof. Integrals $a_k(\lambda)$ can be considered as Fourier coefficients of the function $e^{\lambda s}f(s)$, thus, they converge to zero when $k \to \infty$. It remains to prove that they converge uniformly on the set L_0. The last means that for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for any $|k| \geq n$ and for any $\lambda \in L_0$ we have $|a_k(\lambda)| < \varepsilon$.

Let us suppose the contrary: $\exists \varepsilon > 0$ such that $\forall n \in \mathbb{N}$, $\exists |k| \geq n$, $\exists \lambda \in L_0$: $|a_k(\lambda)| \geq \varepsilon$. Thus, there exists a sequence $k_1 < k_2 < \ldots$ and a sequence $\{\lambda_{k_i}\}_{i=1}^\infty$ such that $|a_{k_i}(\lambda_{k_i})| \geq \varepsilon$.

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Since L_0 is a bounded set then there exists a converging subsequence of $\{\lambda_{k_i}\}_{i=1}^{\infty}$ which we denote by $\{\lambda_j\}_{j\in J}$, where $J \subset N$ is a strictly increasing sequence. Moreover, since L_0 is also closed, then the limit of $\{\lambda_j\}_{j\in J}$ belongs to L_0: $\lambda_j \to \lambda_0 \in L_0$. Let us show that the sequence $\{a_k(\lambda_0)\}$ does not converge to zero.

Indeed, choosing big enough $n \in \mathbb{N}$, such that for any $j > n$, $j \in J$ and any $s \in [-1, 0]$: $|e^{\lambda_0 s} - e^{\lambda_j s}| \leq \frac{\varepsilon}{2}\|f(s)\|$, we obtain

$$|a_j(\lambda_0) - a_j(\lambda_j)| = \left| \int_{-1}^0 e^{2\pi i j s}(e^{\lambda_0 s} - e^{\lambda_j s})f(s)\,\mathrm{d}s \right| \leq \int_{-1}^0 |e^{\lambda_0 s} - e^{\lambda_j s}| f(s)\,\mathrm{d}s \leq \frac{\varepsilon}{2}.$$

Since $|a_j(\lambda_j)| \geq \varepsilon$ and assuming that $|a_j(\lambda_0)| \leq |a_j(\lambda_j)|$, we obtain

$$\frac{\varepsilon}{2} \geq |a_j(\lambda_0) - a_j(\lambda_j)| \geq |a_j(\lambda_j)| - |a_j(\lambda_0)| \geq \varepsilon - |a_j(\lambda_0)|,$$

and, thus, $|a_j(\lambda_0)| \geq \frac{\varepsilon}{2}$ for any $j \in J$, $j > n$.

Thus, $\{a_k(\lambda_0)\}$ does not converge to zero and we have obtained a contradiction with the fact that they are the coefficients of the Fourier series of the function $e^{\lambda_0 s} f(s)$. The last completes the proof of the proposition. $\qquad\square$

Corollary 4.7. *If the sequence $\{\lambda_k\}$ is such that $\mathrm{Im}\lambda_k \to \infty$ and $-\infty < a \leq \mathrm{Re}\lambda_k \leq b < \infty$ then for any $f(s) \in L_2(0, 1; \mathbb{C}^{n\times m})$ we have: $\int_{-1}^0 e^{\lambda_k s} f(s)\,\mathrm{d}s \to 0$ when $k \to \infty$.*

Lemma 4.8. *The following estimates hold:*

1. There exists a constant $C > 0$ such that $\left\|\frac{1}{\lambda}\triangle(\lambda)\right\| \leq C$ for all $\lambda \in \{\lambda : \mathrm{Re}\lambda \geq 0\}\backslash U(0)$, and $\|\triangle(\lambda)\| \leq C$ for all $\lambda \in U(0)$.

2. There exists a constant $C > 0$ such that $\left\|\frac{1}{\lambda}D(z, \xi, \lambda)\right\| \leq C\|(z, \xi(\cdot))\|_{M_2}$ for all $\lambda \in \{\lambda : \mathrm{Re}\lambda \geq 0\}\backslash U(0)$, and $\|D(z, \xi, \lambda)\| \leq C\|(z, \xi(\cdot))\|_{M_2}$ for all $\lambda \in U(0)$, $(z, \xi(\cdot)) \in M_2$.

Proof. From the explicit form (2.14) of $\triangle(\lambda)$ we have an estimate

$$\left\|\frac{1}{\lambda}\triangle(\lambda)\right\| \leq 1 + \|A_{-1}\| + \left\|\int_{-1}^0 e^{\lambda s} A_2(s)\mathrm{d}s\right\| + \frac{1}{|\lambda|}\left\|\int_{-1}^0 e^{\lambda s} A_3(s)\mathrm{d}s\right\|$$

for $\lambda \in \{\lambda : \mathrm{Re}\lambda \geq 0\}\backslash U(0)$, and, thus, it remains to prove that there exists a constant $C_1 > 0$ such that $\left\|\int_{-1}^0 e^{\lambda s} A_i(s)\mathrm{d}s\right\| \leq C_1$, $i = 2, 3$. Indeed, if we suppose the contrary, then there exists an unbounded sequence $\{\lambda_j\}_{j=1}^{\infty}$ such that $\left\|\int_{-1}^0 e^{\lambda_j s} A_i(s)\mathrm{d}s\right\| \to \infty$ when $j \to \infty$. On the other hand, it is easy to see that for any $k \geq 0$: $\int_{-1}^0 e^{\lambda s} s^k \mathrm{d}s \to 0$ when $|\lambda| \to \infty$ and $\lambda \in \{\lambda : \mathrm{Re}\lambda \geq 0\}$. Since the set of polynomials is everywhere dense in $L_2(-1, 0)$, then $\left\|\int_{-1}^0 e^{\lambda s} A_i(s)\mathrm{d}s\right\| \to 0$ when $|\lambda| \to \infty$, $\lambda \in \{\lambda : \mathrm{Re}\lambda \geq 0\}$ and we have come to a contradiction.

The estimate $\|\triangle(\lambda)\| \leq C$ for all $\lambda \in U(0)$ follows easily from the explicit form (2.14).

The estimates for $D(z, \xi, \lambda)$ may be checked directly in the same manner, taking into account that $e^{-\lambda}\int_{-1}^0 e^{-\lambda s} s^k \mathrm{d}s \to 0$ when $|\lambda| \to \infty$ and $\lambda \in \{\lambda : \mathrm{Re}\lambda \geq 0\}$, $k \geq 0$. $\qquad\square$

Now we pass over to the proofs of the main propositions mentioned in the beginning of the section.

Proof of Lemma 4.2. Let us introduce the following notation:

$$f(\lambda) \overset{\mathrm{def}}{=} \triangle^{-1}(\lambda)D(z, \xi, \lambda) = \left(\frac{1}{\lambda}\triangle(\lambda)\right)^{-1}\left(\frac{1}{\lambda}D(z, \xi, \lambda)\right), \quad (z, \xi(\cdot)) \in M_2^0. \tag{4.82}$$

We analyze the behavior of the vector-function $f(\lambda)$ near the imaginary axis. For the points $\lambda_m^k \in \Lambda_1$, which are the eigenvalues of the operator $\mathcal{A}$, the inverse to the matrix $\triangle(\lambda_m^k)$ does not exists. These eigenvalues approach to the imaginary axis when $k \to \infty$. Our first aim is to prove that $f(\lambda)$ is bounded in each neighborhood $U(\lambda_m^k)$ of $\lambda_m^k \in \Lambda_1$, i.e. that the limit $\lim\limits_{\lambda \to \lambda_m^k} \triangle^{-1}(\lambda)D(z,\xi,\lambda)$ exists for all $(z,\xi(\cdot)) \in M_2^0$.

Since $\triangle(\lambda)$ and $D(z,\xi,\lambda)$ are analytic and since, by the construction, all eigenvalues $\lambda_m^k \in \Lambda_1$ are simple, then we have that if λ_m^k is a pole of $f(\lambda)$ then it is a simple pole. In other words, in every neighborhood $U(\lambda_m^k)$ the vector-function $f(\lambda)$ may be represented as follows:

$$f(\lambda) = \frac{1}{\lambda - \lambda_m^k}f_{-1} + \sum_{i=0}^{\infty}(\lambda - \lambda_m^k)^i f_i. \tag{4.83}$$

Thus, our aim is to prove that for each λ_m^k the coefficient $f_{-1} = \lim\limits_{\lambda \to \lambda_m^k}(\lambda - \lambda_m^k)f(\lambda)$ is equal to zero in the representation (4.83), i.e. that $f(\lambda)$ is analytic. To prove this, we construct a representation of the matrix $\left(\frac{1}{\lambda}\triangle(\lambda)\right)^{-1}$ which separates the singularity of this matrix.

According to Lemma 3.5, for each $\lambda_m^k \in \Lambda_1$ there exist matrices $P_{m,k}$, $Q_{m,k}$ such that the value of the matrix-function $\widehat{\triangle}_{m,k}(\lambda) = \frac{1}{\lambda}P_{m,k}R_m\triangle(\lambda)R_mQ_{m,k}$ at the point $\lambda = \lambda_m^k$ has the form (3.54), i.e.

$$\widehat{\triangle}_{m,k}(\lambda_m^k) = \frac{1}{\lambda_m^k}P_{m,k}R_m\triangle(\lambda_m^k)R_mQ_{m,k} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & S_{m,k} & \\ 0 & & & \end{pmatrix}, \quad \det S_{m,k} \neq 0.$$

We rewrite the representation (4.82) of the function $f(\lambda)$ in a neighborhood $U(\lambda_m^k)$ as follows:

$$\begin{aligned} f(\lambda) &= \left(\tfrac{1}{\lambda}R_mP_{m,k}^{-1}P_{m,k}R_m\triangle(\lambda)R_mQ_{m,k}Q_{m,k}^{-1}R_m\right)^{-1}\left(\tfrac{1}{\lambda}D(z,\xi,\lambda)\right) \\ &= R_mQ_{m,k}\left(\tfrac{1}{\lambda}P_{m,k}R_m\triangle(\lambda)R_mQ_{m,k}\right)^{-1}P_{m,k}R_m\left(\tfrac{1}{\lambda}D(z,\xi,\lambda)\right) \\ &= R_mQ_{m,k}\left(\widehat{\triangle}_{m,k}(\lambda)\right)^{-1}P_{m,k}R_m\left(\tfrac{1}{\lambda}D(z,\xi,\lambda)\right). \end{aligned} \tag{4.84}$$

Let us consider the Taylor expansion of the analytic matrix-function $\widehat{\triangle}_{m,k}(\lambda)$ in $U(\lambda_m^k)$:

$$\widehat{\triangle}_{m,k}(\lambda) = \widehat{\triangle}_{m,k}(\lambda_m^k) + (\lambda - \lambda_m^k)\widehat{\triangle}'_{m,k}(\lambda_m^k) + \sum_{i=2}^{\infty}\frac{1}{i!}(\lambda - \lambda_m^k)^i\widehat{\triangle}^{(i)}_{m,k}(\lambda_m^k). \tag{4.85}$$

Due to Corollary (3.8), $\widehat{\triangle}_{m,k}(\lambda)$ allows the representation (3.61) in some $U(\lambda_m^k)$, i.e.

$$\widehat{\triangle}_{m,k}(\lambda) = \begin{pmatrix} (\lambda - \lambda_m^k)r_{11}^{m,k}(\lambda) & (\lambda - \lambda_m^k)r_{12}^{m,k}(\lambda) & \dots & (\lambda - \lambda_m^k)r_{1n}^{m,k}(\lambda) \\ (\lambda - \lambda_m^k)r_{21}^{m,k}(\lambda) & & & \\ \vdots & & S_{m,k}(\lambda) & \\ (\lambda - \lambda_m^k)r_{n1}^{m,k}(\lambda) & & & \end{pmatrix},$$

where $r_{ij}^{m,k}(\lambda)$ are analytic functions, and we note that $S_{m,k}(\lambda_m^k) = S_{m,k}$, where $S_{m,k}$ is defined by (3.54). Differentiating the last relation by λ at $\lambda = \lambda_m^k$, we obtain:

$$\widehat{\triangle}'_{m,k}(\lambda_m^k) = \begin{pmatrix} r_{11}^{m,k}(\lambda_m^k) & r_{12}^{m,k}(\lambda_m^k) & \dots & r_{1n}^{m,k}(\lambda_m^k) \\ r_{21}^{m,k}(\lambda_m^k) & & & \\ \vdots & & S'_{m,k}(\lambda_m^k) & \\ r_{n1}^{m,k}(\lambda_m^k) & & & \end{pmatrix} = \Gamma_{m,k}^0 + \Gamma_{m,k}^1,$$

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$$\Gamma^0_{m,k} \overset{\text{def}}{=} \begin{pmatrix} r^{m,k}_{11}(\lambda^k_m) & r^{m,k}_{12}(\lambda^k_m) & \cdots & r^{m,k}_{1n}(\lambda^k_m) \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix}, \quad \Gamma^1_{m,k} \overset{\text{def}}{=} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ r^{m,k}_{21}(\lambda^k_m) & & & \\ \vdots & & S'_{m,k}(\lambda^k_m) & \\ r^{m,k}_{n1}(\lambda^k_m) & & & \end{pmatrix}.$$

We introduce the matrix-function $F_{m,k}(\lambda) \overset{\text{def}}{=} \widehat{\triangle}_{m,k}(\lambda^k_m) + (\lambda - \lambda^k_m)\Gamma^0_{m,k}$, which has the following structure:

$$F_{m,k}(\lambda) = \begin{pmatrix} r^{m,k}_{11}(\lambda^k_m)(\lambda - \lambda^k_m) & r^{m,k}_{12}(\lambda^k_m)(\lambda - \lambda^k_m) & \cdots & r^{m,k}_{1n}(\lambda^k_m)(\lambda - \lambda^k_m) \\ 0 & & & \\ \vdots & & S_{m,k} & \\ 0 & & & \end{pmatrix}. \quad (4.86)$$

The matrix $F_{m,k}(\lambda)$ is non-singular in a neighborhood $U(\lambda^k_m)\backslash\{\lambda^k_m\}$. Indeed, due to Lemma 3.5 and Corollary 3.8 we have that $\det S_{m,k} \neq 0$, $r^{m,k}_{11}(\lambda^k_m) \neq 0$, and, thus

$$\det F_{m,k}(\lambda) = r^{m,k}_{11}(\lambda^k_m)(\lambda - \lambda^k_m)\det S_{m,k} \neq 0, \quad \lambda \in U(\lambda^k_m)\backslash\{\lambda^k_m\}.$$

Therefore, there exists the inverse matrix $F^{-1}_{m,k}(\lambda)$, which is of the following form:

$$F^{-1}_{m,k}(\lambda) = \begin{pmatrix} \frac{1}{r^{m,k}_{11}(\lambda^k_m)(\lambda - \lambda^k_m)} & F^{m,k}_{21} & \cdots & F^{m,k}_{n1} \\ 0 & F^{m,k}_{22} & \cdots & F^{m,k}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & F^{m,k}_{2n} & \cdots & F^{m,k}_{nn} \end{pmatrix}, \quad (4.87)$$

where

$$\begin{aligned} F^{m,k}_{i1} &= \frac{1}{r^{m,k}_{11}(\lambda^k_m)\det S_{m,k}}\sum_{j=2}^{n}(-1)^{i+j}r^{m,k}_{1j}(\lambda^k_m)[S_{m,k}(\lambda^k_m)]_{ij}, & i = 2,\ldots,n, \\ F^{m,k}_{ij} &= (-1)^{i+j}[S_{m,k}(\lambda^k_m)]_{ij}, & i,j = 2,\ldots,n, \end{aligned} \quad (4.88)$$

and by $[S_{m,k}(\lambda)]_{ij}$ we denote the complementary minor of the element $s^{m,k}_{ij}(\lambda)$, $i,j = 2,\ldots,n$ of the matrix $S_{m,k}(\lambda)$. Since the matrix-functions $S_{m,k}(\lambda)$ are analytic and since $S_{m,k}(\lambda^k_m) \to S$ when $k \to \infty$, then $\|S_{m,k}(\lambda)\|$, $\|[S_{m,k}(\lambda)]_{ij}\|$ and $|s^{m,k}_{ij}(\lambda)|$ are uniformly bounded for all k and $\lambda \in U_\delta(\widetilde{\lambda}^k_m)$. Thus, we conclude that $|F^{m,k}_{ij}| \leq C$ for all k and $i,j = 2,\ldots,n$. Moreover, since $r^{m,k}_{1j}(\lambda^k_m) \to 0$ due to Remark 3.9, then $F^{m,k}_{i1} \to 0$, $i = 2,\ldots,n$ when $k \to \infty$.

Let us rewrite the representation (4.85) as follows:

$$\begin{aligned} \widehat{\triangle}_{m,k}(\lambda) &= F_{m,k}(\lambda) + (\lambda - \lambda^k_m)\Gamma^1_{m,k} + \sum_{i=2}^{\infty}\frac{1}{i!}(\lambda - \lambda^k_m)^i\widehat{\triangle}^{(i)}_{m,k}(\lambda^k_m) \\ &= F_{m,k}(\lambda)\left(I + (\lambda - \lambda^k_m)F^{-1}_{m,k}(\lambda)\Gamma^1_{m,k} + \sum_{i=2}^{\infty}\frac{1}{i!}(\lambda - \lambda^k_m)^i F^{-1}_{m,k}(\lambda)\widehat{\triangle}^{(i)}_{m,k}(\lambda^k_m)\right) \end{aligned} \quad (4.89)$$

and introduce the notation

$$\Upsilon_{m,k}(\lambda) \overset{\text{def}}{=} (\lambda - \lambda^k_m)F^{-1}_{m,k}(\lambda)\Gamma^1_{m,k} + \sum_{i=2}^{\infty}\frac{1}{i!}(\lambda - \lambda^k_m)^i F^{-1}_{m,k}(\lambda)\widehat{\triangle}^{(i)}_{m,k}(\lambda^k_m).$$

Let us prove that for any $\varepsilon > 0$ there exist $\delta > 0$ and $N \in \mathbb{N}$ such that for any $k : |k| > N$ the following estimate holds:

$$\|\Upsilon_{m,k}(\lambda)\| \leq \varepsilon, \quad \lambda \in U_\delta(\widetilde{\lambda}^k_m). \quad (4.90)$$

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From (4.89) we have that $\Upsilon_{m,k}(\lambda) = F_{m,k}^{-1}(\lambda)\widehat{\triangle}_{m,k}(\lambda) - I$, and we are proving the estimate $\|F_{m,k}^{-1}(\lambda)\widehat{\triangle}_{m,k}(\lambda) - I\| \leq \varepsilon$, $\lambda \in U_\delta(\widetilde{\lambda}_m^k)$. Using the representations (3.61), (4.87) and (4.88), we estimate the elements $\{\gamma_{ij}^{m,k}(\lambda)\}_{i,j=1}^n$ of the matrix $\Upsilon_{m,k}(\lambda)$. For the sake of convenience, we divide these elements onto several groups and we begin with the element $\gamma_{11}^{m,k}(\lambda)$:

$$\gamma_{11}^{m,k}(\lambda) = \frac{r_{11}^{m,k}(\lambda)}{r_{11}^{m,k}(\lambda_m^k)} + (\lambda - \lambda_m^k)\sum_{i=2}^n F_{i1}^{m,k} r_{i1}^{m,k}(\lambda) - 1.$$

Due to Corollary 3.8 and since $r_{11}^{m,k}(\lambda)$ is analytic, there exist $\delta > 0$ and $N \in \mathbb{N}$ such that $\left|\frac{r_{11}^{m,k}(\lambda)}{r_{11}^{m,k}(\lambda_m^k)}\right| < \frac{\varepsilon}{2n}$ for all $k : |k| > N$ and $\lambda \in U_\delta(\widetilde{\lambda}_m^k)$. Besides, since $F_{i1}^{m,k} \to 0$, $i = 2,\ldots,n$ and $r_{i1}^{m,k}(\lambda_m^k) \to 0$ when $k \to \infty$, we obtain the estimate $\left|\gamma_{11}^{m,k}(\lambda)\right| < \frac{\varepsilon}{n}$ for all $k : |k| > N$, $\lambda \in U_\delta(\widetilde{\lambda}_m^k)$.

Let us consider other diagonal elements of the matrix $\Upsilon_{m,k}(\lambda)$:

$$\gamma_{jj}^{m,k}(\lambda) = \sum_{i=2}^n F_{ij}^{m,k} s_{ij}^{m,k}(\lambda) - 1 = \sum_{i=2}^n (-1)^{i+j}[S_{m,k}(\lambda)]_{ij}(s_{ij}^{m,k}(\lambda) - s_{ij}^{m,k}(\lambda_m^k)), \quad j = 2,\ldots,n.$$

There exists $\delta > 0$ such that $\left|\gamma_{jj}^{m,k}(\lambda)\right| < \frac{\varepsilon}{n}$ for all $k : |k| > N$, $\lambda \in U_\delta(\widetilde{\lambda}_m^k)$. Further, we consider the elements of the first row:

$$\gamma_{1j}^{m,k}(\lambda) = \frac{r_{1j}^{m,k}(\lambda)}{r_{11}^{m,k}(\lambda_m^k)} + \sum_{i=2}^n F_{i1}^{m,k} s_{ij}^{m,k}(\lambda), \quad j = 2,\ldots,n.$$

Since $F_{i1}^{m,k} \to 0$, $i = 2,\ldots,n$ and $r_{1j}^{m,k}(\lambda_m^k) \to 0$ when $k \to \infty$, we obtain the estimate $\left|\gamma_{1j}^{m,k}(\lambda)\right| < \frac{\varepsilon}{n}$ for all $k : |k| > N$, $\lambda \in U_\delta(\widetilde{\lambda}_m^k)$.

Finally we consider all other elements:

$$\gamma_{ij}^{m,k}(\lambda) = \sum_{r=2}^n F_{ri}^{m,k} s_{rj}^{m,k}(\lambda) = \sum_{r=2}^n (-1)^{i+r}[S_{m,k}(\lambda)]_{ir}(s_{rj}^{m,k}(\lambda) - s_{rj}^{m,k}(\lambda_m^k)), \; i,j = 2,\ldots,n; i \neq j.$$

They may be estimated as $\left|\gamma_{ij}^{m,k}(\lambda)\right| < \frac{\varepsilon}{n}$ for all $k : |k| > N$, $\lambda \in U_\delta(\widetilde{\lambda}_m^k)$ choosing small enough $\delta > 0$.

Finally, we obtain the estimate (4.90) and, therefore, there exist $\delta > 0$, $N \in \mathbb{N}$ such that the matrix $I + \Upsilon_{m,k}(\lambda)$ has an inverse for any $\lambda \in U_\delta(\widetilde{\lambda}_m^k)$, $k : |k| > N$:

$$(I + \Upsilon_{m,k}(\lambda))^{-1} = I + (\lambda - \lambda_m^k)\Gamma_{m,k}(\lambda), \tag{4.91}$$

where $\Gamma_{m,k}(\lambda)$ is analytic in a neighborhood $U_\delta(\lambda_m^k)$.

Finally, from (4.84), (4.89) and (4.91) we obtain:

$$\begin{aligned}
f(\lambda) &= R_m Q_{m,k} \widehat{\triangle}_{m,k}^{-1}(\lambda) P_{m,k} R_m \left(\tfrac{1}{\lambda} D(z,\xi,\lambda)\right) \\
&= R_m Q_{m,k} \left(F_{m,k}(\lambda)(I + \Upsilon_{m,k}(\lambda))\right)^{-1} R_m P_{m,k} \left(\tfrac{1}{\lambda} D(z,\xi,\lambda)\right) \\
&= R_m Q_{m,k} \left(I + (\lambda - \lambda_m^k)\Gamma_{m,k}(\lambda)\right) F_{m,k}^{-1}(\lambda) P_{m,k} R_m \left(\tfrac{1}{\lambda} D(z,\xi,\lambda)\right).
\end{aligned} \tag{4.92}$$

Let us use the fact that $(z,\xi(\cdot)) \in M_2^0$. In Lemma 4.3 the important relation $D(z,\xi,\lambda_m^k) \in \text{Im}\triangle(\lambda_m^k)$, $(z,\xi(\cdot)) \in M_2^0$ is stated. Thus, due to Proposition 4.5, we conclude that

$$\frac{1}{\lambda_m^k} P_{m,k} R_m D(z,\xi,\lambda_m^k) \in \text{Im}\widehat{\triangle}_{m,k}(\lambda_m^k). \tag{4.93}$$

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Moreover, since the matrix $\widehat{\triangle}_{m,k}(\lambda_m^k)$ is of the form (3.54), we conclude that the first component of the vector $\frac{1}{\lambda_m^k}P_{m,k}R_mD(z,\xi,\lambda_m^k)$ equals to zero:

$$\frac{1}{\lambda_m^k}P_{m,k}R_mD(z,\xi,\lambda_m^k) \overset{\text{def}}{=} \widehat{d}_{m,k} = (0,c_2,\ldots,c_n)^T, \tag{4.94}$$

and since the vector-function $\frac{1}{\lambda}P_{m,k}R_mD(z,\xi,\lambda)$ is analytic in a neighborhood $U(\lambda_m^k)$, we conclude that

$$\frac{1}{\lambda}P_{m,k}R_mD(z,\xi,\lambda) = \widehat{d}_{m,k} + d_{m,k}(\lambda), \quad d_{m,k}(\lambda_m^k) = 0. \tag{4.95}$$

Finally, we note that $F_{m,k}^{-1}(\lambda)\widehat{d}_{m,k}$ is a constant vector and the vector-function $F_{m,k}^{-1}(\lambda)d_{m,k}(\lambda)$ is bounded in $U(\lambda_m^k)$. Taking into account (4.92), (4.87) and (4.95), we obtain that

$$
\begin{aligned}
\lim_{\lambda\to\lambda_m^k}(\lambda - \lambda_m^k)f(\lambda) &= R_mQ_{m,k}\lim_{\lambda\to\lambda_m^k}(\lambda - \lambda_m^k)F_{m,k}^{-1}(\lambda)P_{m,k}R_m\left(\tfrac{1}{\lambda}D(z,\xi,\lambda)\right) \\
&= R_mQ_{m,k}\lim_{\lambda\to\lambda_m^k}(\lambda - \lambda_m^k)F_{m,k}^{-1}(\lambda)(\widehat{d}_{m,k} + d_{m,k}(\lambda)) \\
&= 0.
\end{aligned}
\tag{4.96}
$$

Thus, we have proved that $f(\lambda) = \triangle^{-1}(\lambda)D(z,\xi,\lambda)$ is an analytic vector-function in $U_\delta(\lambda_m^k)$, $k : |k| > N$, $m = 1,\ldots,\ell_1$ what gives the estimate $\|\triangle^{-1}(\lambda)D(z,\xi,\lambda)\| \le C_{m,k}$, $(z,\xi(\cdot)) \in M_2^0$.

It remains to prove that $f(\lambda)$ is uniformly bounded in the neighborhoods $U_\delta(\lambda_m^k)$ for all $k : |k| > N$, $m = 1,\ldots,\ell_1$. In other words, our next aim is to prove that the set of vectors

$$f_0 = f_0^{m,k} = f(\lambda_m^k) = (\triangle^{-1}(\lambda)D(z,\xi,\lambda))_{\lambda=\lambda_m^k}$$

is bounded. Taking into account the representation (4.92), we obtain:

$$
\begin{aligned}
f_0^{m,k} &= \left(R_mQ_{m,k}F_{m,k}^{-1}(\lambda)P_{m,k}R_m\left(\tfrac{1}{\lambda}D(z,\xi,\lambda)\right)\right)_{\lambda=\lambda_m^k} \\
&= R_mQ_{m,k}F_{m,k}^{-1}(\lambda)(\widehat{d}_{m,k} + d_{m,k}(\lambda))_{\lambda=\lambda_m^k} \\
&= R_mQ_{m,k}\left(\frac{d_1^1}{r_{11}^{m,k}(\lambda_m^k)} + \sum_{i=2}^n c_iF_{i1}^{m,k},\ \sum_{i=2}^n c_iF_{i2}^{m,k},\ldots,\ \sum_{i=2}^n c_iF_{in}^{m,k}\right)^T,
\end{aligned}
\tag{4.97}
$$

where $d_1^1 = (d'_{m,k}(\lambda_m^k))_1$ is the first component of the derivative of $d_{m,k}(\lambda)$ at the point λ_m^k.

As we have mentioned above, there exists a constant $C_1 > 0$ such that $\|F_{ij}^{m,k}\| \le C_1$, for all $k \in \mathbb{N}$, $m = 1,\ldots,\ell_1$. The estimates $\|P_{m,k}\| \le C_1$ and $\|Q_{m,k}\| \le C_1$ follow from the estimate (3.55) of Lemma 3.7. The estimates $|c_i| < C_1$ and $|d_1^1| < C_1$ follow immediately from Lemma 4.8. From the relation (3.62) of the Corollary 3.8 it follows that there exists a constant $C_2 > 0$ such that $0 < C_2 \le |r_{11}^{m,k}(\lambda_m^k)|$ for all $k \in \mathbb{N}$, $m = 1,\ldots,\ell_1$ and, thus,

$$\frac{1}{|r_{11}^{m,k}(\lambda_m^k)|} \le \frac{1}{C_2}.$$

Finally, we conclude that $\|f_0^{m,k}\| \le C$ for all $m = 1,\ldots,\ell_1$, $k : |k| \ge N$, what completes the proof of the lemma. $\qquad\square$

Proof of Lemma 4.3. Since $g \in M_2^0$ then $g \perp \psi_m^k = \psi(\overline{\lambda_m^k})$ for all $\lambda_m^k \in \Lambda_1$. Therefore, the proposition follows from Lemma 2.9. $\qquad\square$

Proof of Theorem 4.1. Let $\delta > 0$ be such that Lemma 4.2 holds. We divide the closed right half-plane onto the following two sets:

$$K_1(\delta) = \{\lambda : \operatorname{Re}\lambda \geq 0, \ \lambda \in U_\delta(\widetilde{\lambda}_m^k), \ \lambda_m^k \in \Lambda_1\}$$
$$K_2(\delta) = \{\lambda : \operatorname{Re}\lambda \geq 0\}\backslash K_1. \tag{4.98}$$

First, let us estimate $\|R(\lambda, \mathcal{A})x\|$ for any $\lambda \in K_1(\delta)$ and $x \in M_2^0$. Due to Lemma 4.2 we have: $\|\triangle^{-1}(\lambda)D(z,\xi,\lambda)\| \leq C_1\|x\|$, $x = (z, \xi(\cdot)) \in M_2^0$. Due to Corollary 4.7 we have the estimate $\|\int_{-1}^0 e^{-\lambda s}\xi(s)\,ds\| \leq C_2\|x\|$. Thus, for any $x = (z, \xi(\cdot)) \in M_2^0$, $\lambda \in K_1(\delta)$ we obtain:

$$
\begin{aligned}
\|R(\lambda, \mathcal{A})x\| &= \left\| e^{-\lambda}A_{-1}\int_{-1}^0 e^{-\lambda s}\xi(s)\,ds + (I - e^{-\lambda}A_{-1})\triangle^{-1}(\lambda)D(z,\xi,\lambda)\right\|_{\mathbb{C}^n} \\
&\quad + \left\| \int_0^\theta e^{\lambda(\theta-s)}\xi(s)\,ds + e^{\lambda\theta}\triangle^{-1}(\lambda)D(z,\xi,\lambda)\right\|_{L_2} \\
&\leq e^\delta\|A_{-1}\|C_2\|x\| + (1 + e^\delta\|A_{-1}\|)C_1\|x\| \\
&\quad + \left(\int_{-1}^0 \left\| \int_0^\theta e^{\lambda(\theta-s)}\xi(s)\,ds + e^{\lambda\theta}\triangle^{-1}(\lambda)D(z,\xi,\lambda)\right\|_{\mathbb{C}^n}^2\,d\theta\right)^{\frac{1}{2}} \\
&\leq \left[e^\delta\|A_{-1}\|C_2 + (1 + e^\delta\|A_{-1}\|)C_1 + (e^\delta C_2 + C_1^2)^{\frac{1}{2}}\right]\|x\| = C\|x\|.
\end{aligned}
\tag{4.99}
$$

Let us consider $\lambda \in K_2(\delta)$. There exists $\varepsilon > 0$ such that $\left|\frac{1}{\lambda}\det\triangle(\lambda)\right| \geq \varepsilon$ for any $\lambda \in K_2(\delta)\backslash U(0)$. Indeed, if we suppose the contrary then there exists a sequence $\{\lambda_i\}_{i=1}^\infty$ such that $\left|\frac{1}{\lambda_i}\det\triangle(\lambda_i)\right| \to 0$, $i \to \infty$. If the sequence $\{\lambda_i\}_{i=1}^\infty$ is bounded, then it possesses a converging subsequence: $\lambda_{i_j} \to \widehat{\lambda}$ and, thus, $\left|\frac{1}{\widehat{\lambda}}\det\triangle(\widehat{\lambda})\right| = 0$. However, the closure of the set $K_2(\delta)\backslash U(0)$ does not contain zeroes of the function $\det\triangle(\lambda)$ and we have obtained a contradiction.

If the sequence $\{\lambda_i\}_{i=1}^\infty$ is unbounded, i.e. $|\lambda_i| \to \infty$ when $i \to \infty$, then we have $\int_{-1}^0 e^{\lambda_i s}A_2(s)\,ds \to 0$ and $\frac{1}{\lambda_i}\int_{-1}^0 e^{\lambda_i s}A_3(s)\,ds \to 0$, $i \to \infty$. Moreover, since $|\det(-I + e^{-\lambda}A_{-1})| = |\prod(1 + e^{-\lambda}\mu_m)| \geq \prod|1 + e^\delta\mu_m|$, we conclude that

$$\left|\frac{1}{\lambda}\det\triangle(\lambda_i)\right| = \det\left(-I + e^{-\lambda_i}A_{-1} + \int_{-1}^0 e^{\lambda_i s}A_2(s)\,ds + \frac{1}{\lambda_i}\int_{-1}^0 e^{\lambda_i s}A_3(s)\,ds\right) \not\to 0, \quad i \to \infty.$$

Thus, we have obtained a contradiction again.

Taking into account the estimates $\|\frac{1}{\lambda}\triangle(\lambda)\| \leq C_1$ and $\|\frac{1}{\lambda}D(z,\xi,\lambda)\| \leq C_2\|x\|$ from Lemma 4.8, we conclude that $\|\triangle^{-1}(\lambda)D(z,\xi,\lambda)\| \leq C_3\|x\|$ for all $\lambda \in K_2(\delta)\backslash U(0)$. It is easy to see that $\|e^{-\lambda}\int_{-1}^0 e^{-\lambda s}\xi(s)\,ds\| \leq C_4\|x\|$ for all $\lambda \in K_2(\delta)\backslash U(0)$. Finally, similarly to (4.99) we obtain the following estimate

$$\|R(\lambda, \mathcal{A})x\| \leq C\|x\|, \quad \lambda \in K_2.$$

The last completes the proof of the theorem. $\qquad\square$

5 Stability analysis

Basing on the results from Section 3 and Section 4, we prove the main result on stability which does not assume the condition $\det A_{-1} \neq 0$.

Theorem 5.1 (on stability). *If $\sigma(\mathcal{A}) \subset \{\lambda : \text{Re}\lambda < 0\}$ and $\sigma_1 = \sigma(A_{-1}) \cap \{\mu : |\mu| = 1\}$ consists only of simple eigenvalues, then system (1.7) is strongly asymptotically stable.*

Proof. Let us show that $\|e^{t\mathcal{A}}x\| \to 0$ when $t \to +\infty$ for any $x \in M_2$. Due to Theorem 3.3 each $x \in M_2$ allows the following representation:

$$x = x_0 + x_1, \quad x_0 \in M_2^0, \ x_1 \in M_2^1,$$

where M_2^0 and M_2^1 are defined by (3.31)–(3.33) Moreover, the basis of M_2^1 consists of the following eigenvectors:

$$\{\varphi_m^k : \ (\mathcal{A} - \lambda_m^k I)\varphi_m^k = 0, \ \lambda_m^k \in \Lambda_1 = \Lambda_1(N)\}. \tag{5.100}$$

Thus, for any $x_1 \in M_2^1$ we have the representation

$$x_1 = \sum_{m=1}^{\ell_1} \sum_{|k| \geq N} c_m^k \varphi_m^k, \qquad e^{t\mathcal{A}}x_1 = \sum_{m=1}^{\ell_1} \sum_{|k| \geq N} e^{\lambda_m^k t} c_m^k \varphi_m^k, \qquad \sum_{m=1}^{\ell_1} \sum_{|k| \geq N} |c_m^k|^2 < \infty.$$

Let us consider a norm $\| \cdot \|_1$ in which the Riesz basis (5.100) is orthogonal, then we have the following estimate:

$$\|e^{t\mathcal{A}}x_1\|_1 = \left(\sum_{m=1}^{\ell_1} \sum_{|k| \geq N} e^{2\text{Re}\lambda_m^k t} \|c_m^k \varphi_m^k\|_1^2 \right)^{\frac{1}{2}} \leq \|x_1\|_1. \tag{5.101}$$

Since the series $\sum_{m=1}^{\ell_1} \sum_{|k| \geq N} c_m^k \varphi_m^k$ converges and since $\|\varphi_m^k\|_1 \leq C$ for all k and $m = 1, \ldots, \ell_1$, then for any $\varepsilon > 0$ there exists $N_1 \geq N$ such that $\sum_{m=1}^{\ell_1} \sum_{|k| \geq N_1} \|c_m^k \varphi_m^k\|_1^2 \leq \frac{\varepsilon^2}{8}$. Moreover, since the set $\{(m, k) : \ m = 1, \ldots, \ell_1, \ N \leq |k| \leq N_1\}$ is finite and since $\text{Re}\lambda_m^k < 0$, then there exists $t_0 > 0$ such that for any $t \geq t_0$ we have: $\sum_{m=1}^{\ell_1} \sum_{N \leq |k| \leq N_1} e^{2\text{Re}\lambda_m^k t} \|c_m^k \varphi_m^k\|_1^2 \leq \frac{\varepsilon^2}{8}$.

Thus, we obtain

$$\sum_{m=1}^{\ell_1} \sum_{|k| \geq N} e^{2\text{Re}\lambda_m^k t} \|c_m^k \varphi_m^k\|_1^2 \leq \sum_{m=1}^{\ell_1} \sum_{N \leq |k| \leq N_1} e^{2\text{Re}\lambda_m^k t} \|c_m^k \varphi_m^k\|_1^2 + \sum_{m=1}^{\ell_1} \sum_{|k| \geq N_1} \|c_m^k \varphi_m^k\|_1^2 \leq \frac{\varepsilon^2}{4}. \tag{5.102}$$

Due to Theorem 4.1 the semigroup $e^{t\mathcal{A}}|_{M_2^0}$ is exponentially stable, i.e. by definition there exist some positive constants M, ω such that $\|e^{t\mathcal{A}}|_{M_2^0}\| \leq Me^{-\omega t}$. Thus, for any $x_0 \in M_2^0$ there exists $t_0 > 0$ such that for any $t \geq t_0$ we have an estimate

$$\|e^{t\mathcal{A}}x_0\|_1 \leq Me^{-\omega t}\|x_0\|_1 \leq \frac{\varepsilon}{2}. \tag{5.103}$$

Finally, from the estimates (5.101), (5.102) and (5.103) we conclude that for any $x \in M_2$ and for any $\varepsilon > 0$ there exists $t_0 > 0$ such that for any $t \geq t_0$ the following estimate holds:

$$\|e^{t\mathcal{A}}x\|_1 \leq \|e^{t\mathcal{A}}x_0\|_1 + \|e^{t\mathcal{A}}x_1\|_1 \leq \varepsilon. \tag{5.104}$$

Therefore, $\lim_{t \to +\infty} \|e^{t\mathcal{A}}x\|_1 = 0$, what implies that the system (1.6) is strongly asymptotically stable. $\square$

5.1 An example of dilemma: stable and unstable situations

In this subsection we give an explicit example illustrating the item (iii) of Theorem 1.1. Namely, we construct two systems having the same spectrum and satisfying the following conditions: $\sigma(\mathcal{A}) \subset \{\lambda : \mathrm{Re}\lambda < 0\}$ and there are no Jordan blocks, corresponding to eigenvalues from $\sigma_1 = \sigma(A_{-1}) \cap \{\mu : |\mu| = 1\}$, but there exists an eigenvalue $\mu \in \sigma_1$ whose eigenspace is at least two dimensional. Moreover, one of the constructed systems appears to be stable while the other is unstable.

We consider the system of the form

$$\dot{z}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \dot{z}(t-1) + \begin{pmatrix} -b & s \\ 0 & -b \end{pmatrix} z(t), \qquad z \in \mathbb{C}^2, \ t \geq 0, \tag{5.105}$$

where b is a real positive number and for the value of s we essentially distinguish two cases: $s = 0$ and $s \neq 0$.

Remark 5.2. *The systems of the form* $\dot{z} = A_{-1}\dot{z}(t-1) + A_0 z(t)$ *is a special case of the systems (1.7).*

Proof. Indeed, we choose $A_2(\theta) = (\theta + 1)A_0$ and $A_3(\theta) = A_0$. Thus,

$$\int_{-1}^{0} A_2(\theta)\dot{z}(t+\theta) \, \mathrm{d}\theta + \int_{-1}^{0} A_3(\theta)z(t+\theta) \, \mathrm{d}\theta = \int_{-1}^{0} (\theta+1)A_0\dot{z}(t+\theta) \, \mathrm{d}\theta + \int_{-1}^{0} A_0 z(t+\theta) \, \mathrm{d}\theta$$

$$= A_0 \int_{-1}^{0} ((\theta+1)z(t+\theta))' \, \mathrm{d}\theta = A_0 z(t).$$

$\square$

The eigenvalues of the operator $\mathcal{A}$ are the roots of the equation $\det \triangle_{\mathcal{A}}(\lambda) = 0$, which, in our particular case, has the form:

$$\det(-\lambda I + \lambda e^{-\lambda}A_{-1} + A_0) = \det \begin{pmatrix} -\lambda - \lambda e^{-\lambda} - b & s \\ 0 & -\lambda - \lambda e^{-\lambda} - b \end{pmatrix} = 0.$$

Thus, all the eigenvalues of the operator $\mathcal{A}$ satisfy the equation

$$\lambda e^\lambda + \lambda + b e^\lambda = 0 \tag{5.106}$$

and the multiplicity of each eigenvalue equals two. To prove that $\sigma(\mathcal{A}) \subset \{\lambda : \mathrm{Re}\lambda < 0\}$, we use the results on transcendental equations obtained by L. Pontryagin [17].

Let us consider the equation $H(z) = 0$, where $H(z) = h(z, e^z)$ is a polynomial with respect to z and e^z.

Definition 5.3. *We say that the function* $H(z) = \sum\limits_{m,n} a_{mn} z^m e^{nz}$ *possesses the principal term* $a_{rs} z^r e^{sz}$, *if for all other terms* $a_{mn} z^m e^{nz}$ *we have that* $r \geq m$ *and* $s \geq n$.

We denote by $F(y) : \mathbb{R} \to \mathbb{R}$ and $G(y) : \mathbb{R} \to \mathbb{R}$, correspondingly the real and the imaginary parts of the function $H(iy)$, i.e. $H(iy) = F(y) + iG(y)$, $y \in \mathbb{R}$.

Definition 5.4. *We say that the zeroes of two real-valued functions of a real variable alternate if and only if*
a) each of these functions has no multiple roots;
b) between any two zeroes of one of these functions there exists at least one zero of the other;
c) the functions are never simultaneously zero.

Theorem 5.5 ([17, Pontryagin, 1942]). *Let $H(z) = h(z, e^z)$ be a polynomial with a principal term.*

1. If all the zeroes of the function $H(z)$ belong to the open left half-plane: $\operatorname{Re}\lambda_k < 0$, then the zeroes of $F(y)$ and $G(y)$ are real, alternating and for all $y \in \mathbb{R}$ the following inequality holds

$$G'(y)F(y) - G(y)F'(y) > 0. \tag{5.107}$$

2. Any of the conditions below is sufficient for all the zeroes of the function $H(z)$ to lie in the open left half-plane:

a) all the zeroes of the functions $F(y)$ and $G(y)$ are real, alternating and the inequality (5.107) is satisfied for at least one value of y;

b) all the zeroes of the function $F(y)$ are real and for each zero $y = y_0$ the inequality (5.107) is satisfied, i.e. $G(y_0)F'(y_0) < 0$;

c) all the zeroes of the function $G(y)$ are real and for each zero $y = y_0$ the inequality (5.107) is satisfied, i.e. $G'(y_0)F(y_0) > 0$.

The following theorem gives a criterion for all zeroes of a function to be real.

Theorem 5.6 ([17, Pontryagin, 1942]). *Let $F(z) = f(z, \cos z, \sin z)$ be a polynomial with a principal term $z^r \phi_m^{(s)}(\cos z, \sin z)$, where $\phi_m^{(s)}(\cos z, \sin z)$ is homogeneous with respect to $\cos z$ and $\sin z$ polynomial.*

The function $F(z)$, $z \in \mathbb{C}$ possesses only real zeroes if and only if for all big enough $k \in \mathbb{Z}$ the function $F(x)$, $x \in \mathbb{R}$ possesses exactly $4ks + r$ real roots on the interval $-2\pi k + \varepsilon \le x \le 2\pi k + \varepsilon$ for some $\varepsilon > 0$.

We use the results mentioned above to analyze the location of the roots of the equation (5.106).

Proposition 5.7. *For any $b > 0$ and any $s \in \mathbb{C}$ the spectrum of the corresponding operator $\mathcal{A}$ belongs to the open left half-plane.*

Proof. We represent $H(iy) = iye^{iy} + iy + be^{iy}$ as follows:

$$
\begin{aligned}
H(iy) &= iy(\cos y + i \sin y) + iy + b(\cos y + i \sin y) \\
&= (b \cos y - y \sin y) + i(y \cos y + y + \sin y) \\
&= F(y) + iG(y).
\end{aligned}
$$

Since the roots of the equations $y = 0$ and $\cos y = 0$ are not roots of the equation $F(y) = 0$, we rewrite $F(y) = 0$ as $\operatorname{tg} y = \frac{b}{y}$. Several zeroes of the equation $F(y) = 0$ may be seen on the figure below:
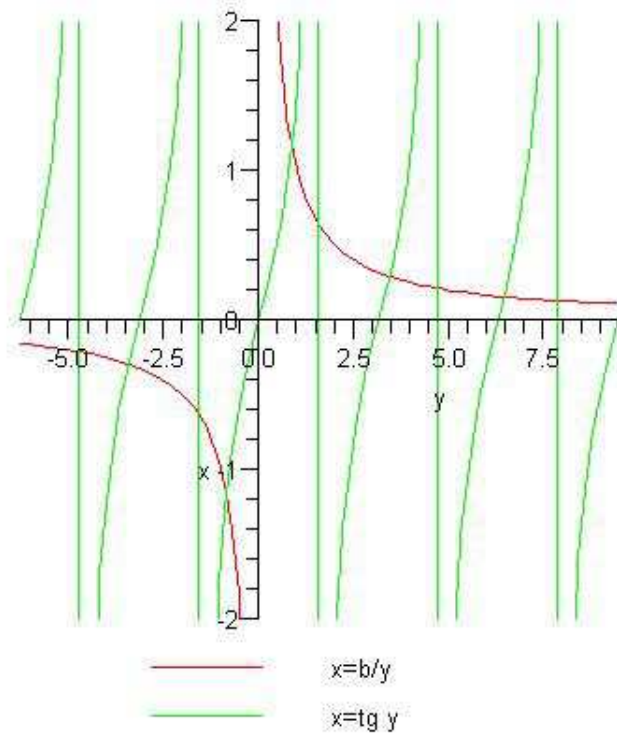
Figure 3.

We see that exactly 5 zeroes of the equation $F(y) = 0$ belong to the interval $-2\pi + \varepsilon \leq y \leq 2\pi + \varepsilon$ if we choose $\varepsilon > 0$ "big" enough such that the zero on the right from 2π belongs to the interval. Adding to this interval from the left and from the right 2π we add each time 4 other zeroes. Thus, on each interval $-2\pi k + \varepsilon \leq y \leq 2\pi k + \varepsilon$ we have exactly $4k + 1$ zeroes of the equation $F(y) = 0$ and, therefore, the conditions of the Theorem 5.6 are satisfied. Thus, all the zeroes of the equation $F(y) = 0$ are real.

Let us now prove that for each root y_0 of the equation $F(y) = 0$ the inequality $G(y_0)F'(y_0) < 0$ holds. We use the notation $\cos y = C$, $\sin y = S$:

$$-GF' = -(-bS - S - y_0C)(y_0C + y_0 + bS) = (bS + S + y_0C)(y_0C + y_0 + bS).$$

We rewrite $\mathrm{tg}\, y_0 = \frac{b}{y_0}$ as $C = \frac{Sb}{y_0}$ and substitute C at the last relation:

$$-GF' = \left(bS + \frac{y_0^2}{b}S + S\right)\left(\frac{y_0^2}{b}S + bS + y_0\right) = S^2\left(b + \frac{y_0^2}{b} + 1\right)\left(\frac{y_0^2}{b} + b + \frac{y_0}{S}\right).$$

Since $S^2 > 0$ and $b + \frac{y_0^2}{b} + 1 > 0$, then it remains to prove, that the third multiplier is greater then zero. From $\mathrm{tg}\, y_0 = \frac{b}{y_0}$ we also have $b = \frac{y_0 S}{C}$ and substituting we obtain:

$$\frac{y_0^2}{b} + b + \frac{y_0}{S} = \frac{y_0 C}{S} + \frac{y_0 S}{C} + \frac{y_0}{S} = y_0\frac{C^2 + S^2 + C}{SC} = y_0\frac{1 + C}{SC} = (1 + C)bC^2 > 0.$$

We note also that the fact that $y_0 SC > 0$ can be easily seen from the picture above.

and use the sufficient condition b) of Theorem 5.6.

Applying Theorem 5.6 we complete the proof of the fact that all the eigenvalues of the operator $\mathcal{A}$ belongs to the open left half-plane. $\qquad\square$

Proposition 5.8. *If $s = 0$ then the operator $\mathcal{A}$ possesses eigenvectors only, i.e. it possesses no root vectors; if $s \neq 0$ then to each eigenvalue $\lambda \in \sigma(\mathcal{A})$ there corresponds a pair of eigenvector and root vector of the operator $\mathcal{A}$.*

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Proof. If $s = 0$ then $\Delta_{\mathcal{A}}(\lambda_k)$ is the zero-matrix for each eigenvalue λ_k, and, therefore, the the the space of solutions of the equation $\Delta_{\mathcal{A}}(\lambda_k)z = 0$ is two-dimensional. We choose the following basis of this space: $z_k^1 = (1,0)^T$, $z_k^2 = (0,1)^T$.

Let us consider the equation for eigenvectors:

$$(\mathcal{A} - \lambda_k I)\begin{pmatrix} y \\ z(\theta) \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} A_0 z(0) - \lambda_k y \\ \dot{z}(\theta) - \lambda_k z(\theta) \end{pmatrix} = 0.$$

The solution of the second equation is given by $z(\theta) = e^{\lambda_k \theta} z(0)$, thus, $z(-1) = e^{-\lambda_k} z(0)$. Taking into account the domain of the operator $\mathcal{A}$: $y = z(0) - A_{-1}z(-1) = (I - e^{-\lambda_k} A_{-1})z(0)$, we obtain from the first equation: $(A_0 - \lambda_k I + \lambda_k e^{-\lambda_k} A_{-1})z(0) = 0$, or, in an equivalent form: $\Delta_{\mathcal{A}}(\lambda_k)z(0) = 0$.

As we have noted above the last equation has two-dimensional solution: $z(0) = z_k^1 = (1,0)^T$ and $z(0) = z_k^2 = (0,1)^T$, and therefore, there is a two-dimensional eigenspace of the operator $\mathcal{A}$ corresponding to the eigenvalue λ_k. Two eigenvectors of this subspace can be chosen in the following form: $f_k^1 = \begin{pmatrix} y_k^1 \\ z_k^1(\cdot) \end{pmatrix}$, where $y_k^1 = \begin{pmatrix} 1 + e^{-\lambda_k} \\ 0 \end{pmatrix}$, $z_k^1(\theta) = \begin{pmatrix} e^{\lambda_k \theta} \\ 0 \end{pmatrix}$

and $f_k^2 = \begin{pmatrix} y_k^2 \\ z_k^2(\cdot) \end{pmatrix}$, where $y_k^2 = \begin{pmatrix} 0 \\ 1 + e^{-\lambda_k} \end{pmatrix}$, $z_k^2(\theta) = \begin{pmatrix} 0 \\ e^{\lambda_k \theta} \end{pmatrix}$.

Thus, to any eigenvalue λ_k of the operator $\mathcal{A}$ there corresponds the two-dimensional eigenspace.

If $s = 1$ (or $s \neq 0$) we have that $\Delta_{\mathcal{A}}(\lambda_k) = \begin{pmatrix} 0 & e^{-\lambda_k} \\ 0 & 0 \end{pmatrix}$ and, therefore, the space of solutions of the equation $\Delta_{\mathcal{A}}(\lambda_k)z = 0$ is one-dimensional: $z_k^1 = (1,0)^T$ (since, obviously, $e^{-\lambda_k} \neq 0$). Thus, the equation $\Delta_{\mathcal{A}}(\lambda_k)z = z_k^1$ has also one-dimensional solution which we denote by $z_k^2 = (0,1)^T$.

Arguing as above we show that the operator $\mathcal{A}$ possesses one eigenvector corresponding to the eigenvalue λ_k: $f_k^1 = \begin{pmatrix} y_k^1 \\ z_k^1(\cdot) \end{pmatrix}$, where $y_k^1 = \begin{pmatrix} 1 + e^{-\lambda_k} \\ 0 \end{pmatrix}$, $z_k^1(\theta) = \begin{pmatrix} e^{\lambda_k \theta} \\ 0 \end{pmatrix}$.

Let us show that the operator $\mathcal{A}$ possesses one root vector. Each root vector f of the operator $\mathcal{A}$ satisfies the following equation: $(\mathcal{A} - \lambda_k I)^2 f = 0$, i.e. $\widetilde{f} = (\mathcal{A} - \lambda_k I)f \in$ $\mathrm{Ker}(\mathcal{A} - \lambda_k I)$. From the last we conclude that $\widetilde{f} = f_k^1 = \begin{pmatrix} (1 + e^{-\lambda_k}, 0)^T \\ (e^{\lambda_k \theta}, 0)^T \end{pmatrix}$ and the root vector satisfies the relation

$$\begin{pmatrix} A_0 z(-1) - \lambda_k y \\ \dot{z}(\theta) - \lambda_k z(\theta) \end{pmatrix} = \begin{pmatrix} (1 + e^{-\lambda_k}, 0)^T \\ (e^{\lambda_k \theta}, 0)^T \end{pmatrix}.$$

The solution of the second equation is given by

$$z(\theta) = e^{\lambda_k \theta} z(0) + \int_0^\theta e^{\lambda_k (\theta - \tau)} e^{\lambda_k \tau} y_k^1 \, d\tau = e^{\lambda_k \theta} z(0) + \theta e^{\lambda_k \theta} y_k^1,$$

what implies $z(-1) = e^{-\lambda_k} z(0) - e^{-\lambda_k} y_k^1$. Taking into account the domain of the operator $\mathcal{A}$, we write down the first equation in the following form:

$$A_0 z(0) - \lambda_k \left(z(0) - A_{-1}(e^{-\lambda_k} z(0) - e^{-\lambda_k} y_k^1) \right) = (1 + e^{-\lambda_k}) y_k^1$$

which is equivalent to

$$\Delta_{\mathcal{A}}(\lambda_k)z(0) = (1 + e^{-\lambda_k}) y_k^1 + \lambda_k A_{-1} e^{-\lambda_k} y_k^1 = \begin{pmatrix} 1 + e^{-\lambda_k} - \lambda_k e^{-\lambda_k} \\ 0 \end{pmatrix}.$$

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Thus, if $1 + e^{-\lambda_k} - \lambda_k e^{-\lambda_k} \neq 0$ then $z(0)$ is the root vector of the matrix $\Delta_A(\lambda_k)$ and, therefore, $z(0) = z_k^2 = (0,1)^T$ and the operator $\mathcal{A}$ possesses the root vector $f_k^2 = \begin{pmatrix} y_k^2 \\ z_k^2(\cdot) \end{pmatrix}$,

where $y_k^2 = \begin{pmatrix} e^{-\lambda_k} \\ 1 + e^{-\lambda_k} \end{pmatrix}$ and $z_k^2(\theta) = \begin{pmatrix} \theta e^{\lambda_k \theta} \\ e^{\lambda_k \theta} \end{pmatrix}$.

It remains to show that $1 + e^{-\lambda_k} - \lambda_k e^{-\lambda_k} \neq 0$. We suppose the contrary: $\lambda_k e^{-\lambda_k} = 1 + e^{-\lambda_k}$ and, multiplying the last expression onto e^{λ_k}, we obtain $\lambda_k = e^{\lambda_k} + 1$. Since λ_k is an eigenvalue of $\mathcal{A}$, then we obtain $e^{\lambda_k}(e^{\lambda_k} + 1) + e^{\lambda_k} + 1 + e^{\lambda_k} = 0$ or $e^{2\lambda_k} + 3e^{\lambda_k} + 1 = 0$, and we conclude that $e^{\lambda_k} = \frac{-3 \pm \sqrt{5}}{2}$. For the root $e^{\lambda_k} = \frac{-3 - \sqrt{5}}{2}$ we have that $\mathrm{Re}\lambda_k > 0$, and for the root $e^{\lambda_k} = \frac{-3 + \sqrt{5}}{2}$ we have: $\lambda_k = \frac{-3 + \sqrt{5}}{2} + 1 = \frac{\sqrt{5} - 1}{2} > 0$. We have obtained the contradiction what completes the analysis of eigenvectors and root vectors of the operator $\mathcal{A}$. $\qquad\square$

Proposition 5.9. *If $s = 0$ then the system (5.105) is strongly asymptotically stable; if $s \neq 0$ then the system is unstable.*

Proof. First we consider the system (5.105) when $s = 0$ and prove that in this case the system is stable. Let us evaluate the norm of eigenvectors.

$$\|f_k^1\|^2 = \|f_k^2\|^2 = (1 + e^{-\lambda_k})(\overline{1 + e^{-\lambda_k}}) + \int_{-1}^{0} e^{\lambda_k \theta} \overline{e^{\lambda_k \theta}}\, d\theta = |1 + e^{-\lambda_k}|^2 + \int_{-1}^{0} e^{2\mathrm{Re}\lambda_k \theta}\, d\theta$$

$$= |1 + e^{-\lambda_k}|^2 + \frac{1}{2\mathrm{Re}\lambda_k}\left(1 - e^{-2\mathrm{Re}\lambda_k}\right).$$

Since $\mathrm{Re}\lambda_k \to 0$ when $k \to \infty$, then $\lim\limits_{k \to \infty} \frac{1}{2\mathrm{Re}\lambda_k}\left(1 - e^{-2\mathrm{Re}\lambda_k}\right) = 1$. Therefore, $0 < C_1 \leq \frac{1}{2\mathrm{Re}\lambda_k}\left(1 - e^{-2\mathrm{Re}\lambda_k}\right) \leq C_2$. Taking into account that $|e^{-\lambda_k}| \leq C_3$, we obtain the following estimates:

$$C_1 \leq \|f_k^i\|^2 \leq (1 + C_3)^2 + C_2 = C_4. \tag{5.108}$$

As it has been shown in [21], the subspaces $V^{(k)} = \mathrm{Lin}\{f_k^1, f_k^2\}$ (and the finite-dimensional subspace W_N) form a Riesz basis of the space M_2. Since we have proved the estimate (5.108) then the eigenvectors $\{f_k^1, f_k^2\}$ (together with vectors from W_N) form a basis of M_2. Therefore, we have a Riesz basis of eigenvectors. Further we just repeat the proof given in [21, Theorem 23].

We consider a norm $\|\cdot\|_1$ in which the eigenvectors $\{f_k^1, f_k^2\}_{k \in \mathbb{Z}}$ are orthogonal. Let a vector x belongs to a closed span of the subspaces $V^{(k)}$, then $x = \sum\limits_{k \in \mathbb{Z}}(\alpha_k f_k^1 + \beta_k f_k^2)$ and we have:

$$e^{\mathcal{A}t}x = \sum_{k \in \mathbb{Z}} e^{\lambda_k t}(\alpha_k f_k^1 + \beta_k f_k^2).$$

Therefore,

$$\|e^{\mathcal{A}t}x\|_1^2 = \sum_{k \in \mathbb{Z}} e^{\lambda_k t}(\|\alpha_k f_k^1\|_1^2 + \|\beta_k f_k^2\|_1^2) \leq \sum_{k \in \mathbb{Z}}(\|\alpha_k f_k^1\|_1^2 + \|\beta_k f_k^2\|_1^2) = \|x\|_1^2$$

and, thus, the family $e^{\mathcal{A}t}$ is uniformly bounded in the subspace generated by the subspaces $V^{(k)}$. From the last we conclude that the system is strongly asymptotically stable.

Let us consider the system (5.105) when $s = 1$.

Let an operator $\mathcal{A}$ has a sequence of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ such that $\mathrm{Re}\lambda_k < 0$ and $\mathrm{Re}\lambda_k \to 0$ when $k \to \infty$ and to each λ_k there corresponds one eigenvector v^k and at least

one root vector w^k. We show that the equation $\dot{x} = \mathcal{A}x$ is unstable. Let us suppose that $\|v^k\| = \|w^k\| = 1$. Since, for each w^k we have $e^{\mathcal{A}t}w^k = e^{\lambda_k t}(tv^k + w^k)$ then

$$\|e^{\mathcal{A}t}w^k\| = |e^{\lambda_k t}|\|(tv^k + w^k)\| \geq e^{\text{Re}\lambda_k t}(t-1).$$

For any constant $C > 0$ we take $t \geq 2C + 1$ and for this t we take big enough k such that $e^{\text{Re}\lambda_k t} \geq \frac{1}{2}$. Then we have:

$$\|e^{\mathcal{A}t}w^k\| \geq \frac{1}{2}(2C + 1 - 1) = C$$

and we conclude that $\|e^{\mathcal{A}t}\| \geq C$ for $t \geq 2C + 1$. Therefore, the family of exponents $e^{\mathcal{A}t}$ is not uniformly bounded and because of Banach-Steinhaus theorem there exists $x \in D(A)$ such that $\|e^{\mathcal{A}t}x\| \to \infty$ when $t \to +\infty$.

Thus, the system (5.105) is unstable when $s = 1$. The last completes the proof of the proposition. $\qquad\square$

6 Stabilizability analysis

In this section we prove Theorem 1.2 for control systems (1.8) with $\det A_{-1} = 0$. It is convenient to reformulate this theorem as follows.

Theorem 6.1 (on stabilizability). *Let $b_1, \ldots, b_p \in \mathbb{C}^n$ be the columns of the matrix B. Assume that the following four conditions are satisfied.*

(1) *All the eigenvalues of the matrix A_{-1} satisfy $|\mu| \leq 1$.*

(2) *All the eigenvalues $\mu \in \sigma_1$ are simple.*

(3) *$\sum_{i=1}^{p} |\langle b_i, y \rangle_{\mathbb{C}^n} \neq 0$ for all $i = 1, \ldots, p$ and all vectors y satisfying $y \in \text{Ker}\triangle_{\mathcal{A}}^*(\lambda)$ for roots λ of the equation $\det \triangle_{\mathcal{A}}^*(\lambda) = 0$, such that $\text{Re}\lambda \geq 0$.*

(4) *$\sum_{i=1}^{p} |\langle b_i, y_m \rangle_{\mathbb{C}^n} \neq 0$ for all vectors y_m of the matrix A_{-1}^*, corresponding to eigenvalues $\overline{\mu}_m$, $\mu_m \in \sigma_1$ and for all $i = 1, \ldots, p$.*

Then there exists a regular control $u = \mathcal{F}x$ of the form

$$u = \mathcal{F}x = \int_{-1}^{0} F_2(\theta)\dot{z}_t(\theta)\,\mathrm{d}\theta + \int_{-1}^{0} F_3(\theta)z_t(\theta)\,\mathrm{d}\theta \tag{6.109}$$

where $x = (y, z(\cdot)) \in \mathcal{D}(\mathcal{A})$, $F_2(\cdot), F_3(\cdot) \in L_2(-1, 0; \mathbb{C}^{n \times p})$. And this control stabilizes the system (1.8), i.e. $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A} + \mathcal{B}\mathcal{F})$ and $e^{t(\mathcal{A} + \mathcal{B}\mathcal{F})}x_0 \to 0$ as $t \to \infty$ for all $x_0 \in M_2$.

The controllability conditions (3)–(4) of Theorem 6.1 are equivalent to (3)–(4) of Theorem 1.2 (for more details concerning these conditions see [19]). We also note that regular stabilizability for a particular case of the control systems (1.8) had been considered in [18]. Before giving the proof, let us discuss the conditions (1) and (2). Since the regular feedback does not change the matrix A_{-1} we need the assumption $\sigma(A_{-1}) \subset \{\mu : |\mu| \leq 1\}$. Moreover, taking into account the results of Theorem 5.1 on strong stability, we conclude that by means of a regular feedback it is possible to stabilize the system in the case when the algebraic multiplicity of each eigenvalue $\mu \in \sigma_1$ equals to 1. In this case the closed loop system will be asymptotically stable if and only if all the eigenvectors of $\mathcal{A} + \mathcal{B}\mathcal{F}$ are in the left half-plane. Thus, the problem of regular stabilizability for such systems consists in assigning the spectrum of the system in the left half-plane. The most intensional problem in the situation appears when an infinite number of the eigenvalues located "close" to the imaginary axis belongs to the right half-plane. This means, in particular, that $\sigma_1 \neq \emptyset$.

Proof of Theorem 6.1. Let us construct the decomposition of the spectrum of $\mathcal{A}$ introduced in Section 3.2 by (3.41):
$$\sigma(\mathcal{A}) = \Lambda_0(\mathcal{A}) \cup \Lambda_1(\mathcal{A}) \cup \Lambda_2(\mathcal{A}), \tag{6.110}$$
where the subsets are given by (3.42). Since $\Lambda_0(\mathcal{A}) \subset \{\lambda : \operatorname{Re}\lambda \leq -\varepsilon\}$, $\varepsilon > 0$ then our aim is to construct the feedback which moves the eigenvalues of $\Lambda_1(\mathcal{A})$ and $\Lambda_2(\mathcal{A})$ to the left half-plane.

We begin with moving the spectral set $\Lambda_2(\mathcal{A})$. The spectral decomposition of the state space $M_2 = M_2^0 \oplus M_2^1 \oplus M_2^2$ (Theorem 3.4), corresponding to (6.110), allows us to rewrite the system (1.9) as

$$\begin{cases} \dot{x}_0 = \mathcal{A}_0 x_0 + \mathcal{B}_0 u, & x_0 \in M_2^0 = M_2^0(\mathcal{A}) \\ \dot{x}_1 = \mathcal{A}_1 x_1 + \mathcal{B}_1 u, & x_1 \in M_2^1 = M_2^1(\mathcal{A}) \\ \dot{x}_1 = \mathcal{A}_2 x_2 + \mathcal{B}_2 u, & x_1 \in M_2^1 = M_2^2(\mathcal{A}), \end{cases} \tag{6.111}$$

where $\mathcal{A}_0 = \mathcal{A}|_{M_2^0}$, $\mathcal{A}_1 = \mathcal{A}|_{M_2^1}$, $\mathcal{A}_2 = \mathcal{A}|_{M_2^2}$ and $\mathcal{B}_i$ are the projections of $\mathcal{B}$ onto subspaces M_2^i.

The spectrum $\Lambda_2(\mathcal{A})$ of the operator $\mathcal{A}_2$ defined on the finite-dimensional subspace M_2^2 belongs to the right half-plane. Due to the assumption (3) of the theorem all the eigenvalues of $\mathcal{A}_2$ are controllable and, thus, they may be assigned arbitrary to the half-plane $\{\lambda : \operatorname{Re}\lambda \leq -\varepsilon\}$. Namely, let us consider a feedback $u = \mathcal{F}_2 x$, $\mathcal{F}_2 : M_2 \to \mathbb{C}^p$ which acts as follows:

$$\mathcal{F}_2 x_0 = \mathcal{F}_2 x_1 = 0, \ x_0 \in M_2^0, x_1 \in M_2^1, \quad \mathcal{F}_2 x_2 = \widehat{\mathcal{F}}_2 x_2, \ x_2 \in M_2^2,$$

where $\widehat{\mathcal{F}} : M_2^2 \to \mathbb{C}^p$ is a bounded operator. It is easy to see that such form of feedback allows to move only eigenvalues of $\Lambda_2(\mathcal{A})$, i.e. $\Lambda_0(\mathcal{A}) \cup \Lambda_1(\mathcal{A}) \subset \sigma(\mathcal{A} + \mathcal{B}\mathcal{F}_2)$. Thus, we conclude that there exists $\widehat{\mathcal{F}}$ such that the spectrum of the closed-loop system $\mathcal{A} + \mathcal{B}\mathcal{F}_2$ is of the form
$$\sigma(\mathcal{A} + \mathcal{B}\mathcal{F}_2) = \sigma(\mathcal{A}) \backslash \Lambda_2(\mathcal{A}) \cup \widehat{\Lambda}_2, \quad \widehat{\Lambda}_2 \subset \{\lambda : \operatorname{Re}\lambda \leq -\varepsilon\}. \tag{6.112}$$

We choose the control in the form $u = \mathcal{F}_2 x + v$, denote $\widehat{\mathcal{A}} \overset{\text{def}}{=} \mathcal{A} + \mathcal{B}\mathcal{F}_0$ and rewrite the system (1.9) as
$$\dot{x} = \widehat{\mathcal{A}}x + \mathcal{B}v. \tag{6.113}$$
We emphasize that due to the form of the feedback $\mathcal{F}_2$ the infinite-dimensional system (6.113) corresponds to the neutral type system (1.8) with the same matrix A_{-1}.

Let us now construct the decomposition of the spectrum (3.41) for the operator $\widehat{\mathcal{A}}$. Due to (6.112), we obtain $\Lambda_2(\widehat{\mathcal{A}}) = \emptyset$ and, thus,

$$\sigma(\widehat{\mathcal{A}}) = \Lambda_0(\widehat{\mathcal{A}}) \cup \Lambda_1(\widehat{\mathcal{A}}), \qquad \Lambda_1(\widehat{\mathcal{A}}) = \Lambda_1(\mathcal{A}).$$

Applying Theorem 3.4, we construct the spectral decomposition of the state space $M_2 = M_2^0 \oplus M_2^1$. The operator model (6.113) may be rewritten as follows:

$$\begin{cases} \dot{x}_0 = \widehat{\mathcal{A}}_0 x_0 + \widehat{\mathcal{B}}_0 v, & x_0 \in M_2^0 = M_2^0(\widehat{\mathcal{A}}) \\ \dot{x}_1 = \widehat{\mathcal{A}}_1 x_1 + \widehat{\mathcal{B}}_1 v, & x_1 \in M_2^1 = M_2^1(\widehat{\mathcal{A}}), \end{cases} \tag{6.114}$$

where $\widehat{\mathcal{A}}_0 = \widehat{\mathcal{A}}|_{M_2^0}$, $\widehat{\mathcal{A}}_1 = \widehat{\mathcal{A}}|_{M_2^1}$ and $\widehat{\mathcal{B}}_i$ are the projections of the vectors of $\mathcal{B}$ onto subspaces M_2^i.

Due to Theorem 4.4 the restriction of the resolvent $R(\lambda, \widehat{\mathcal{A}})|_{M_2^0}$ is uniformly bounded on the set $\{\lambda : \mathrm{Re}\lambda \geq 0\}$ and, therefore, the semigroup $\{e^{t\widehat{\mathcal{A}}}|_{M_2^0}\}_{t\geq 0}$ is exponentially stable.

To stabilize the second equation we apply the approach introduced in [22] which is based on the abstract theorem on infinite pole assignment. Below, for the sake of completeness and due to the specific form of the operator $\widehat{\mathcal{A}}_1$, we give a simplified formulation of the mentioned theorem (Theorem 6.5).

The theorem on infinite pole assignment holds for a single input system. However, as it is shown in [22], the multivariable case may be reduced to the single input case by the following considerations (see [22] and also the classical result on finite-dimensional control systems in [30]).

Let us consider the vector $\mathbf{b}$ given by

$$\mathbf{b} = \begin{pmatrix} b \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 b_1 + \ldots + c_p b_p \\ 0 \end{pmatrix} = \begin{pmatrix} B \\ 0 \end{pmatrix} c = \mathcal{B}c, \quad c = \begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix}, \tag{6.115}$$

where $\{b_1, \ldots, b_p\} \subset \mathbb{C}^n$ are the columns of the matrix B. The exist numbers $c_i \in \mathbb{C}$ such that the following relations hold (Lemma 6.6):

(a) $\langle b, y_m \rangle_{\mathbb{C}^n} \neq 0$ for all eigenvectors y_m of the matrix A^*_{-1} corresponding to the eigenvalues $\overline{\mu_m}$, where $\mu_m \in \sigma_1$, $m = 1, \ldots, \ell_1$.

(b) $\langle b, y_m^k \rangle_{\mathbb{C}^n} \neq 0$ for all such y_m^k which satisfy $y_m^k \in \mathrm{Ker}\triangle^*(\overline{\lambda_m^k})$ and $\mathrm{Re}\lambda_m^k \geq 0$.

Let us denote the projection of $\mathbf{b}$ onto the subspace M_2^1 by $\mathbf{b}_1$ and consider the single input system

$$\dot{x}_1 = \widehat{\mathcal{A}}_1 x_1 + \mathbf{b}_1 \widehat{u}, \quad \widehat{u} \in \mathbb{C}. \tag{6.116}$$

Due to the construction, the eigenvalues of $\widehat{\mathcal{A}}_1$ are simple and each eigenvalue λ_m^k, $m = 1, \ldots, \ell_1$, $|k| \geq N_1$ belongs to the circle $L_m^k(r^{(k)})$ centered at $\widetilde{\lambda}_m^k = i(\arg \mu_m + 2\pi k)$. The corresponding eigenvectors φ_m^k form a Riesz basis (Proposition 2.7).

Thus, the single input system (6.116) satisfies the conditions (H1)–(H4) of Theorem 6.5. According to that theorem the spectrum $\Lambda_1(\mathcal{A})$ of the operator $\widehat{\mathcal{A}}_1$ may be moved by the regular feedback (6.109) as follows.

Let us chose scalars $\widehat{\lambda}_m^k$ such that $\mathrm{Re}\widehat{\lambda}_m^k < 0$ and which are located inside the circles $L_m^k(r^{(k)})$. There exists a feedback $\widehat{u} = \widehat{\mathcal{F}}_1 x$, $\widehat{\mathcal{F}}_1 : M_2 \to \mathbb{C}$ such that $\widehat{\lambda}_m^k$ are eigenvalues of the operator $\widehat{\mathcal{A}}_1 + \mathbf{b}_1 \widehat{\mathcal{F}}_1$. Taking into account (6.115), we define the feedback $v = \mathcal{F}_1 x$, $\mathcal{F}_1 : M_2 \to \mathbb{C}^p$ as follows

$$F_1 x \overset{\mathrm{def}}{=} c\widehat{\mathcal{F}}_1 x.$$

Due to the theorem on stability the semigroup $\{e^{t(\widehat{\mathcal{A}}+\mathcal{B}\mathcal{F}_1)}|_{M_2^1}\}_{t\geq 0}$ is asymptotically stable. Thus, the feedback

$$u = \mathcal{F}_1 x_1 + \mathcal{F}_2 x_2$$

transforms the original system into a system where all the conditions of Theorem 5.1 on asymptotic stability are verified. The last completes the proof of Theorem 6.1. $\square$

Remark 6.2. *We would like to emphasize that in the present paper to prove the result on stabilizability for the case $\det A_{-1} = 0$ we have contributed the ideas which are mainly of technical character: the direct decomposition of the state space and the proof of the resolvent boundedness on some subspace. However, the main contribution from the stabilizability point of view is the abstract theorem on infinite pole assignment which had been proved in [22].*

Remark 6.3. *The stabilizability of the restriction of the system onto the subspace $M_2^2(\mathcal{A})$ follows also from some classical results. Let us denote by Γ_δ a rectifiable, simple, closed curve which surround the spectral set $\Lambda_2(\mathcal{A})$ and $\Gamma_\delta \subset \{\lambda : \operatorname{Re}\lambda \geq \varepsilon\}$. By $P_2 = \frac{1}{2\pi i} \int\limits_{\Gamma_\delta} R(\mathcal{A}, \lambda)\, d\lambda$ we denote the spectral projector. The subspaces $P_2 M_2$ and $(I - P_2)M_2$ are $\mathcal{A}$-invariant and the spectrum of the restriction $(I - P_2)M_2$ belongs to the half-plane $\{\lambda : \operatorname{Re}\lambda < \varepsilon\}$. According to [26] the spectral set $\Lambda_2(\mathcal{A})$ may be assigned arbitrary to the half-plane $\{\lambda : \operatorname{Re}\lambda < \varepsilon\}$ by means of a finite rank input operator (under some controllability conditions). The development of this approach was given in [14] and [9] (see e.g. [5] as a survey).*

Remark 6.4. *Let us discuss the assumptions (3)–(4) of Theorem 1.2:*
(3) $\operatorname{rank}(\triangle(\lambda),\ B) = n$ for all $\lambda : \operatorname{Re}\lambda \geq 0$.
(4) $\operatorname{rank}(\mu I - A_{-1},\ B) = n$ for all $\mu \in \sigma_1$.
*   The assumptions (3) may include an infinite number of relations (obviously, it should be verified only for λ which are eigenvalues of $\mathcal{A}$, i.e. $\det\triangle_\mathcal{A}(\lambda) = 0$; there may exist an infinite number of eigenvalues of $\mathcal{A}$ belonging to the right half-plane). However, only a finite number of these relations have to be verified. More precisely, there exists $M > 0$ such that for any eigenvalue λ such that $|\operatorname{Im}\lambda| > M$ the condition (3) follows from the condition (4).*

Proof. Let A_{-1} be of the form (2.16) and we prove the proposition for eigenvalues located inside the circles $L_1^k(r^{(k)})$, i.e. for λ_1^k (for all other indices $m = 1, \ldots, \ell_1$ the idea of the proof remains the same). The matrix $A_{-1} - \mu_1 I = A_{-1} - e^{-\widetilde{\lambda}_1^k} I$ is of the form:

$$A_{-1} - \mu_1 I = \begin{pmatrix} 0 & 0 \\ 0 & \widehat{A} \end{pmatrix}, \qquad \widehat{A} \in \mathbb{C}^{(n-1)\times(n-1)}, \ \det \widehat{A} \neq 0.$$

Due to the assumption (4) there exists $i \in \overline{1, p}$ such that the column $b = (b_{1i}, \ldots, b_{ni})^T$ of thee matrix B possesses a nonzero first component: $b_{1i} \neq 0$. Besides, we note that since $\det \widehat{A} \neq 0$ then there exists the unique $\alpha = (\alpha_2, \ldots, \alpha_n)^T \in \mathbb{C}^{n-1}$ such that $\widehat{b} = \widehat{A}\alpha$, where $\widehat{b} = (b_{2i}, \ldots, b_{ni})^T \in \mathbb{C}^{n-1}$.
   Let us analyze the structure of $\triangle(\lambda_1^k)$, we rewrite it as follows:

$$\triangle(\lambda_1^k) = \lambda_1^k e^{-\lambda_1^k}(A_{-1} - \mu_1 I + D_k),$$

where

$$D_k = (e^{\widetilde{\lambda}_1^k} - e^{\lambda_1^k})I + e^{\lambda_1^k}\int_{-1}^{0} e^{\lambda_1^k \theta}\left(A_2(\theta) + \frac{1}{\lambda_1^k}A_3(\theta)\right) d\theta.$$

From the last formula we conclude, that $D_k \to 0$, when $k \to \infty$.
   Let us show that there exists $N \in \mathbb{N}$ such that for any $k : |k| \geq N$:

$$\operatorname{rank}(\triangle(\lambda_1^k),\ b) = n, \tag{6.117}$$

which is equivalent to the statement of the proposition.
   Since $\det \triangle(\lambda_1^k) = 0$ then $\det(A_{-1} - \mu_1 I + D_k) = 0$. Let us denote by $\widehat{D}_k = \{(D_k)_{ij}\}_{i,j=2}^{n}$. Since $\det \widehat{A} \neq 0$, then there exists $N \in \mathbb{N}$ such that for any $k : |k| \geq N$ we have: $\det(\widehat{A} + \widehat{D}_k) \neq 0$. Therefore, there exists the unique vector $\alpha_k \in \mathbb{C}^n$ such that $\widehat{b} = (\widehat{A} + \widehat{D}_k)\alpha_k$. From the last we conclude

$$\alpha_k = (\widehat{A} + \widehat{D}_k)^{-1}\widehat{b} = (I + \widehat{A}^{-1}\widehat{D}_k)^{-1}\widehat{A}^{-1}\widehat{b} = (I + \widehat{A}^{-1}\widehat{D}_k)^{-1}\alpha.$$

Since $(I + \widehat{A}^{-1}\widehat{D}_k)^{-1} \to I$ when $k \to \infty$, we conclude that $\alpha_k \to \alpha$.

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If we suppose that (6.117) does not hold, then $b_1 = b_{1i}$ allows the following representation:

$$b_{1i} = (\alpha_k)_2 (D_k)_{12} + (\alpha_k)_n (D_k)_{1n}.$$

However, since $(D_k)_{1i} \to 0$, then the right-hand side of the last relation tends to zero when $k \to \infty$. We have come to the contrary, which completes the proof of the proposition. $\square$

As it was mentioned above, for the sake of completeness we give further the formulations of the results on infinite pole assignment from [22]. We formulate them in a form, which takes into account the specific form of the operator $\widehat{\mathcal{A}}_1$.

Theorem 6.5 (On infinite pole assignment [22]). *Let H be a complex Hilbert space, $\mathcal{A}$ be an infinitesimal generator of a C_0-semigroup in H, and the control system if given by $\dot{x} = \mathcal{A}x + \mathcal{B}u$, $x \in \mathcal{D}(\mathcal{A}) \subset H$. Let $\mu_1, \dots, \mu_\ell$ be some nonzero complex numbers and we introduce complex numbers*

$$\widetilde{\lambda}_m^k = \ln |\mu_m| + \mathrm{i}(\arg \mu_m + 2\pi k), \quad m = 1, \dots, \ell, k \in \mathbb{Z},$$

and the circles $L_m^k(r^{(k)})$ centered at $\widetilde{\lambda}_m^k$ with radii $r^{(k)}$ satisfying the relation $\sum\limits_{k \in \mathbb{Z}} (r^{(k)})^2 < \infty$.

Let the following assumptions hold:

(H1) *The spectrum of $\mathcal{A}$ consists only of eigenvalues which are located in the circles $L_m^k(r^{(k)})$. Moreover, all the eigenvalues of $\mathcal{A}$ are simple (i.e. its algebraic multiplicity equals 1) and there exists $N_1 \in \mathbb{N}$ such that for any $k : |k| \geq N_1$ the total multiplicity of the eigenvalues contained in the circles $L_m^k(r^{(k)})$ equals 1.*

(H2) *The corresponding eigenvectors, which we denote by φ_m^k, constitute a Riesz basis in H.*

(H3) *The system $\dot{x} = \mathcal{A}x + \mathcal{B}u$ is of a single input, i.e. the operator $\mathcal{B} : \mathbb{C} \to H$ is the operator of multiplication by $\mathbf{b} \in H$.*

(H4) *We assume the following controllability condition: $\mathbf{b}$ is not orthogonal to eigenvectors ψ_m^k of the operator $\mathcal{A}^*$: $\langle \mathbf{b}, \psi_m^k \rangle \neq 0$ and*

$$\lim_{k \to \infty} k |\langle \mathbf{b}, \psi_m^k \rangle| = c_m, \qquad 0 < c_m < +\infty. \tag{6.118}$$

Then there exists $N_2 \geq N_1$ such that for any family of complex numbers $\widehat{\lambda}_m^k \in L_m^k(r^{(k)})$, $|k| \geq N_2$ there exists a linear control $\mathcal{F} : \mathcal{D}(\mathcal{A}) \to \mathbb{C}$, such that

(1) *the complex numbers $\widehat{\lambda}_m^k$ are eigenvalues of the operator $\mathcal{A} + \mathcal{B}\mathcal{F}$;*

(2) *the operator $\mathcal{B}\mathcal{F} : \mathcal{D}(\mathcal{A}) \to H$ is relatively $\mathcal{A}$-bounded.*

Lemma 6.6 ([22]). *If the assumptions (3) and (4) of Theorem 6.1 holds, then there exists a vector $b \in \mathrm{Im}B$, say $b = c_1 b_1 + \dots + c_p b_p$, $c_i \in \mathbb{C}$, such that the following relations hold:*
(a) $\langle b, y_m \rangle_{\mathbb{C}^n} \neq 0$ for all eigenvectors y_m of the matrix A_{-1}^* corresponding to the eigenvalues $\overline{\mu_m}$, where $\mu_m \in \sigma_1$, $m = 1, \dots, \ell_1$.
(b) $\langle b, y_m^k \rangle_{\mathbb{C}^n} \neq 0$ for all such y_m^k which satisfy $y_m^k \in \mathrm{Ker}\triangle^*(\overline{\lambda_m^k})$ and $\mathrm{Re}\lambda_m^k \geq 0$.

Proof. Let us prove the relations (a). Consider the subspaces

$$L_m = \{y : \langle y, y_m \rangle_{\mathbb{C}^n} = 0\}, \quad m = 1, \ldots, \ell_1.$$

These subspaces are of dimension $n - 1$. Let us consider also $M_m = L_m \cap \mathrm{Im}B$. By the assumption (4) we have: $\dim M_m < p$. Indeed, if $\dim M_m = p = \dim \mathrm{Im}B$, then $B^*y = 0$ and the condition (4) is not satisfied. Thus, each M_m is nowhere dense in $\mathrm{Im}B$ and due to Baire theorem we have:

$$\bigcup_{m=1}^{\ell_1} M_m \neq \mathrm{Im}B.$$

The last means that there exists $b \in \mathrm{Im}B$ such that $b \notin L_m$ for all $m = 1, \ldots, \ell_1$, i.e. the relations $\langle b, y_m \rangle_{\mathbb{C}^n} \neq 0$ hold.

The inequalities (b) follows from the relation

$$|\langle b, y_m^k \rangle_{\mathbb{C}^n}| \sim C_m(|k| + 1)^{-1}$$

which is proved in [22] independently of the condition $\det A_{-1} \neq 0$. $\square$

7 Conclusion and perspectives

In the present paper we have generalized the results on strong asymptotic non-exponential stability and regular stabilizability for the case of mixed retarded-neutral type systems. The proofs of the mentioned generalizations are technically complicated and requires subtle estimates. We combine the Riesz basis technic with the analysis of the boundedness of the resolvent on some $\mathcal{A}$-invariant subspaces.

As a perspective, we consider systems with the difference operator K given by

$$Kf = \sum_{i=1}^{r} A_{h_i} f(h_i), \qquad h_i \in [-1, 0].$$

Besides, the dilemma of item (iii) of Theorem 1.1 on stability may be investigated more precisely.

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